Atti Sem. Mat. Fis. Univ. Modena, Supplemento al Vol. XLVI, 803-825 (1998)

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The Monotone Integral [2]

dedicated to Prof. Calogero Vinti in Honour of His 70th birthday

Abstract: Here we introduce a new definition of the monotone integral, in infinite dimensional setting, in order to obtain the equivalence between the Bochner and monotone integrals.

1 Introduction

Given a measurable space (Ω, Σ) and a measurable non negative function $f : \Omega \to \mathbb{R}_0^+$, the monotone integral of f with respect to a set function m defined on Σ can be defined in terms of the integrability (and previously the measurability) of the function $\phi(t) = m(\{\omega \in \Omega : f(\omega) > t\})$. The monotone integral of a measurable scalar function with respect to a scalar set function m has been widely studied in literature ([5], [11]).

In [4] an extended definition of the monotone integral has been introduced as an alternative way of integrating scalar functions with respect to Banach-valued finitely additive measures. Nevertheless the definition adopted there turned out to be stronger than expected: indeed a counterexample given in the same paper shows that there exist scalar

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[2] Nota giunta in Redazione il 14-V-1997.
Mathematical Subject Classification: 28A70
Lavoro svolto nell' ambito del G.N.A.F.A. del C.N.R.
This paper will be published also in the "*Proceedings of the Conference in Honour of Professor Vinti*", edited by this journal

functions that are integrable in the usual way (namely by approximating via simple functions), but not in the monotone sense.

Since then some attempts to find the "right" definition of monotone integral, namely equivalent to the classical integration, have been done, but in a not totally satisfactory way: in [12] and [13] the equivalence between the strong monotone integral and the classical one has been shown for finitely additive measures ranging in a Hilbert space or in a nuclear space, and under a "nice" condition on the finitely additive measure.
In [1] a definition of the monotone integral for scalar functions with respect to set functions with values in Dedekind complete Riesz spaces is given.

In this paper we introduce a definition for the monotone integral with respect to a Banach-valued finitely additive measure which makes use of the Fremlin-McShane integrability of the function ϕ. Finally, it turns out that this is the right approach in order to obtain the seeken equivalence of the two theories.

2 Notations and Preliminaries

Troughout this paper we shall use the following notations.
_ (Ω, Σ) is a measurable space, where Σ is a σ-algebra.
_ X is a Banach space, X^* is the topological dual of X.
_ X_1 (resp. X_1^*) is the unit ball in X (resp. X^*).
_ λ is the Lebesgue measure on $\mathbb{R}$ and $\mathcal{B}$ is the Borel σ-algebra, $\mathcal{A}$ is the family of open sets of $\mathbb{R}$.
_ $m : \Sigma \to X$ is a strongly bounded finitely additive measure and $\|m\|$ is its semivariation. Since m is strongly bounded, it admits a Rybakov control (see [15]) $\nu = \mid x_0^* m \mid$, with $x_0^* \in X_1^*$.
If $f, f_n : \Omega \to \mathbb{R}_0^+$ are Σ-measurable functions, we define the following upper level functions,

for every $E \in \Sigma$ and for every $t \in \mathbb{R}_0^+$:

$$\begin{array}{ll}
\phi(t) = m(x \in \Omega : f(x) > t); & \phi^E(t) = m(x \in E : f(x) > t); \\
\phi_n(t) = m(x \in \Omega : f_n(x) > t); & \phi_n^E(t) = m(x \in E : f_n(x) > t); \\
\Gamma(t) = \nu(x \in \Omega : f(x) > t); & \Gamma^E(t) = \nu(x \in E : f(x) > t); \\
\Gamma_n(t) = \nu(x \in \Omega : f_n(x) > t); & \Gamma_n^E(t) = \nu(x \in E : f_n(x) > t); \\
\widehat{\phi}(t) = \|m\|(x \in \Omega : f(x) > t); & \widehat{\phi}^E(t) = \|m\|(x \in E : f(x) > t); \\
\widehat{\phi}_n(t) = \|m\|(x \in \Omega : f_n(x) > t); & \widehat{\phi}_n^E(t) = \|m\|(x \in E : f_n(x) > t).
\end{array}$$

Definition 2.1 A *generalized McShane partition* of $\mathbb{R}_0^+$ is a sequence $(T_n, t_n)_{n\in\mathbb{N}}$ of pairwise disjoint measurable sets of finite measure such that $\lambda(\mathbb{R}_0^+ - \bigcup_n T_n) = 0$ and $t_n \in \mathbb{R}_0^+$, for every $n \in \mathbb{N}$.

Definition 2.2 A *gauge* is a function $\Delta : \mathbb{R}_0^+ \to \mathcal{A}$ such that $y \in \Delta(y)$ for every $y \in \mathbb{R}_0^+$.

Definition 2.3 We say that a generalized McShane partition $(T_n, t_n)_n$ is *subordinate* to a gauge Δ if for every $n \in \mathbb{N}, T_n \subset \Delta(t_n)$.

Definition 2.4 An *Henstock partition* of $[0,1]$ is a finite family of non overlapping intervals $([a_i, a_{i+1}], t_i)_{i\le n}$ which covers $[0,1]$ and such that for every $1 \le i \le n$, $t_i \in [a_i, a_{i+1}]$. Given a gauge $\Delta : [0,1] \to \mathcal{A}$ an Henstock partition is *subordinate* to Δ if

$$[a_i, a_{i+1}] \subset \Delta(t_i),$$

for every $i = 1, ..., n$.

Definition 2.5 A *partial Mc Shane partition* of $I\!R_0^+$ is a countable family $(T_n, t_n)_n$ where $(T_n)_n$ is a disjoint family of sets of finite λ-measure, and $t_n \in I\!R_0^+$ for every $n \in \mathbb{N}$; and it is *subordinate* to a gauge Δ if $T_n \subset \Delta(t_n)$ for every n.

Definition 2.6 ([9]) A function $\phi : I\!R_0^+ \to X$ is *McShane-integrable* on $I\!R_0^+$ if there exists $w \in X$ such that for every $\varepsilon > 0$ there exists a gauge $\Delta(\varepsilon) : I\!R_0^+ \to \mathcal{A}$ such that

$$\limsup_{n\to\infty} \left\| w - \sum_{i=1}^{n} \lambda(T_i)\phi(t_i) \right\| \le \varepsilon$$

for every generalized McShane partition $(T_i, t_i)_i$ subordinate to $\Delta(\varepsilon)$.

Definition 2.7 Let $f : \Omega \to \mathbb{R}_0^+$ be a measurable function. We say that f is ($\star$)-*integrable* if, for every $E \in \Sigma$, there exists an element $w^E \in X$, such that for every $\varepsilon > 0$ there exists a gauge $\Delta(\varepsilon) : I\!R_0^+ \to \mathcal{A}$ (the gauge must be the same for every $E \in \Sigma$) such that

$$\limsup_{n\to\infty} \left\| w^E - \sum_{i=1}^{n} \lambda(T_i)\phi^E(t_i) \right\| \leq \varepsilon$$

for every generalized McShane partition $(T_i, t_i)_i$ subordinate to $\Delta(\varepsilon)$, and we set

$$\int_E^\star f dm = w^E.$$

Definition 2.8 Let $f : \Omega \to \mathbb{R}$. We say that f is ($\star$)-integrable iff f^+, f^- are ($\star$)-integrable and we define

$$\int_E^\star f dm = \int_E^\star f^+ dm - \int_E^\star f^- dm.$$

We denote by $L^{\star,1}(m)$ the class of ($\star$)-integrable functions.

Definition 2.9 ([4]) Let $f : \Omega \to \mathbb{R}$ be a measurable function. Then f is m-*integrable* if there exists a sequence of simple functions $(f_n)_n$ such that $(f_n)_n$ ν-converges to f for any control ν for m and the sequence $(\int_F f_n dm)_n$ converges in X for every $F \in \Sigma$. In this case we set

$$\int_{(\cdot)} f dm = \lim_{n\to\infty} \int_{(\cdot)} f_n dm.$$

We denote by $L^1(m)$ the space of m-integrable functions.

If X is separable we can introduce also the following definition of integrability:

Definition 2.10 ([4]) Let $f : \Omega \to \mathbb{R}_0^+$ be a measurable function. Then f is ($\widehat{\ }$)-*integrable* with respect to m if $\widehat{\phi}(t)$ is Lebesgue integrable. In this case $\phi(t)$ is Bochner-integrable and we set

$$\widehat{\int_{(\cdot)}} f dm = \int_0^\infty \phi(t) dt.$$

We denote by $\widehat{L}^1(m)$ the class of ($\widehat{\ }$)-integrable functions.
Observe that, if X is separable, and f is measurable then ϕ is weakly of bounded variation and therefore weakly measurable. By Pettis Theorem [14], ϕ is measurable.

3 Measurability of the distribution functions

Lemma 3.1 *Let $f:\Omega\to\mathbb{R}_0^+$ be a $(\star)$-integrable function. Then*

$$\lim_{t\to\infty}\|m\|(\{\omega\in\Omega : f(\omega)>t\})=0.$$

Proof: Since f is $(\star)$-integrable, then by Proposition 1Q of [9] $f\in L^1(x_0^* m)$, by Lemma 3.5 of [4] $f\in L^1(\nu)$. By Markov inequality it follows that

$$\nu(\{\omega\in\Omega : f(\omega)>t\})\leq\frac{1}{t}\int_\Omega f d\nu.$$

Using the ν-continuity of $\|m\|$ we have

$$\lim_{t\to\infty}\|m\|(\{\omega\in\Omega : f(\omega)>t\})=0.$$

Though the Mc Shane definition of integrability does not request the measurability of the integrand ϕ, Fremlin, in [9], proves that the integrand is weakly measurable. Here we prove that if f is $(\star)$-integrable then ϕ is totally measurable.

Proposition 3.2 *Let $f:\Omega\to\mathbb{R}_0^+$ be a measurable function such that*

$$\lim_{t\to\infty}\|m\|(\{\omega\in\Omega : f(\omega)>t\})=0.$$

Then the function $\phi:\mathbb{R}_0^+\to X$ defined by $\phi(t)=m(f>t)$ is λ-totally measurable.

Proof: Let H be the set of the discontinuity points of $\widehat{\phi}$. Observe that by the monotonicity of the functions $\widehat{\phi}$ and Γ, H is a countable set. Hence $\lambda(H)=0$.
For every $n\in\mathbb{N}$ and for every $k=0,...,n2^n-1$ we set

$$E_{n,k} = \{\omega\in\Omega : \frac{k}{2^n}\leq f(\omega)<\frac{k+1}{2^n}\} \qquad E_{n,n2^n}=\{\omega\in\Omega : f(\omega)\geq n\}.$$

We define

$$f_n(\omega)=\sum_{k=0}^{n2^n-1}\frac{k}{2^n}\cdot 1_{E_{n,k}}(\omega)+0\cdot 1_{E_{n,n2^n}}(\omega).$$

The sequence of simple functions $(f_n)_n$ satisfies the following conditions:

a) $f_n(\omega) \leq f(\omega) \wedge n$, for every $\omega \in \Omega$ and for every $n \in \mathbb{N}$;

b) $\lim_{n\to\infty} f_n(\omega) = f(\omega)$, for every $\omega \in \Omega$;

c) the sets $E_{n,k}$ are pairwise disjoint and $\bigcup_{k=0}^{n2^n} E_{n,k} = \Omega$.

ϕ_n is a simple function and for every $t \in \mathbb{R}_0^+$ we have

$$\phi_n(t) = m\{\omega \in \Omega : \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \cdot 1_{E_{k,n}}(\omega) > t\} = \sum_{k=0}^{n2^n-1} m\{\omega \in E_{n,k} : \frac{k}{2^n} > t\}.$$

and

$$\phi(t) = \sum_{k=0}^{n2^n} m\{\omega \in E_{n,k} : f(\omega) > t\}.$$

Thus, for every $t \in \mathbb{R}_0^+ \setminus H$ and for every $n \in \mathbb{N}$ with $n > t$ there exists $\overline{k}(n,t)$ such that

$$\|\phi(t) - \phi_n(t)\| \leq \|m\|(E_{n,\overline{k}(n,t)}) + \|m\|(E_{n,n2^n}).$$

Setting $a_n = \frac{\overline{k}(n)}{2^n}$ and $b_n = \frac{\overline{k}(n)+1}{2^n}$ we obtain $a_n \leq a_{n+1} \leq t \leq b_{n+1} \leq b_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = t$ and $\lim_{n\to\infty} b_n = t$.
Then

$$\|\phi(t) - \phi_n(t)\| \leq \|m\|(E_{n,\overline{k}(n,t)}) + \|m\|(E_{n,n2^n}) = \|m\|(f^{-1}([a_n, b_n[)) + \|m\|(E_{n,n2^n})$$

By hypothesis $\lim_{n\to\infty} \|m\|(E_{n,n2^n}) = 0$. Since μ is finitely additive,

$$\nu(f^{-1}([a_n, b_n[)) \leq \mu(f \geq a_n) - \nu(f > b_n).$$

We shall prove that $\lim_{n\to\infty} \nu(f \geq a_n) = \nu(f > t)$.
Let $(a_n^{'})_n$ be a non increasing sequence such that for every $n \in \mathbb{N}$, $a_n^{'} \leq a_n$, $a_n^{'} \uparrow t$ and $\nu(f > a_n^{'}) = \nu(f \geq a_n^{'})$.
Obviously

$$\nu(f > t) \leq \nu(f \geq a_n) \leq \nu(f \geq a_n^{'}) = \nu(f > a_n^{'}).$$

By the monotonicity of $\widehat{\phi}$ and since $t \notin H$

$$\lim_{n\to\infty} \nu(f > a_n^{'}) = \nu(f > t)$$

and so the assertion follows.
Thus for every $t \in \mathbb{R}_0^+ \setminus H$ we obtain, since $m \ll \nu$,

$$\lim_{n\to\infty} \|\phi(t) - \phi_n(t)\| = 0.$$

4 Comparison

Proposition 4.1 *Let $f : \Omega \to \mathbb{R}_0^+$ be a simple measurable function. Then f is $(\star)$-integrable and m-integrable and the two integrals coincide.*

Proof: It is enough to prove the result for indicator functions. In fact the Mc Shane integral is additive, see for example [9] 1C. Let $f = x \cdot 1_H$, where $x \in \mathbb{R}_0^+, H \in \Sigma$. Obviously f is m-integrable with $\int_{\cdot} f dm = x \cdot m(\cdot \cap H)$. For every $E \in \Sigma$ we have

$$\phi^E(t) = m(E \cap H) \cdot 1_{[0,x[}(t).$$

For every $\varepsilon > 0$ let F be a closed subset of $[0,x[$ such that $\lambda([0,x[\setminus F) \leq \varepsilon$. We define $\Delta_\varepsilon : \mathbb{R}_0^+ \to \mathcal{A}$ as follows:

$$\Delta_\varepsilon(s) = \begin{cases} [0,x[& \text{if } s \in \text{F} \\ [0,x[\setminus F & \text{if } s \in [0,x[\setminus F \\ I\!R_0^+ \setminus F & \text{if } s \in [x,+\infty[\end{cases}$$

Let $(T_i, t_i)_i$ be a generalized McShane partition of $\mathbb{R}_0^+$ subordinated to Δ_ε.

$$\begin{aligned} & \|x \cdot m(E \cap H) - \sum_{i \leq n} \lambda(T_i)\phi^E(t_i)\| = \|x \cdot m(E\cap H) - \sum_{i\leq n, t_i < x} \lambda(T_i) m(E \cap H)\| = \\ = \; & \|m(E \cap H)\| \cdot |x - \sum_{i \leq n, t_i < x} \lambda(T_i)| \leq \|m\|(\Omega) \cdot |x - \sum_{i\leq n, t_i<x} \lambda(T_i)|. \end{aligned}$$

$(T_i \cap F)_i$ is such that $\lambda(F - \bigcup_{t_i \in F} T_i) = 0$. Since $\bigcup_{t_i < x} T_i \supset \bigcup_{t_i\in F} T_i \supset \bigcup_{t_i \in F}(T_i \cap F)$

$$\begin{aligned} \lim_n \quad & \lambda\left([0,x[\setminus \bigcup_{i\leq n, t_i<x} T_i\right) = \lambda\left([0,x[\setminus \bigcup_{t_i<x} T_i\right) \leq \lambda\left([0,x[\setminus \bigcup_{t_i\in F} T_i\right) \leq \\ \leq \quad & \lambda\left([0,x[\setminus \bigcup_{t_i \in F}(T_i\cap F)\right) = x - \lambda\left(\bigcup_{t_i<x}(T_i \cap F)\right) = x - \lambda(F) \leq \varepsilon. \end{aligned}$$

So the assertion follows.

Now we want to compare $L^1(m)$ and $L^{\star,1}(m)$. To obtain this we need some preliminary results.

Proposition 4.2 *Let $f\Omega \to \mathbb{R}_0^+$ be a measurable function such that*

$$\lim_{t\to\infty} \|m\|(\{\omega \in \Omega : f(\omega) > t\} = 0).$$

Let
$f_n(\omega) = \sum_{k=0}^{k=n2^n-1} \frac{k}{2^n} 1_{E_{n,k}}(\omega)$ *where* $E_{n,k} = \left\{\omega \in \Omega : \frac{k}{2^n} \leq gf(\omega) < \frac{k+1}{2^n}\right\}$.
Then the simple functions ϕ_n, which are the upper level sets of f_n, are Bochner integrable.

Proof: By Proposition 3.2 the functions ϕ_n are totally measurable; since they are simple, they are Bochner integrable.

Proposition 4.3 *Let $\Delta : \mathbb{R}_0^+ \to \mathcal{A}$ be a gauge. Then for every $\varepsilon > 0$ there exists a generalized McShane partition $(E_n, t_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_0^+$ subordinate to Δ such that for every $n \in \mathbb{N}$*

1) $E_n = [a_n, a_{n+1}]$, *where* $a_0 = 0$*;*

2) $t_n \in E_n$*;*

3) $a_{n+1} - a_n < \varepsilon$.

Proof: Let $\varepsilon > 0$. Let $\Delta_n = \Delta\,|_{[\frac{\varepsilon}{2}n, \frac{\varepsilon}{2}(n+1)]}$. Applying Lemma 5 of [6] when $K = A = [\frac{\varepsilon}{2}n, \frac{\varepsilon}{2}(n+1)]$ there exists a partial Henstock partition $([a_i^n, a_{i+1}^n],\ t_i)_{i\leq k(n)}$ subordinate to Δ such that $[a_i^n, a_{i+1}^n] \subset [\frac{\varepsilon}{2}n, \frac{\varepsilon}{2}(n+1)]$, $t_i \in [a_i^n, a_{i+1}^n]$, for every $i = 1, ..., k(n)$ and $[\frac{\varepsilon}{2}n, \frac{\varepsilon}{2}(n+1)] \subset \bigcup_{i\leq k(n)}[a_i^n, a_{i+1}^n]$. Now we consider the family $([a_i^n, a_{i+1}^n], t_i)_{i\leq k(n), n\in\mathbb{N}}$. This is the desired generalized McShane partition of $\mathbb{R}_0^+$.

Theorem 4.4 *Let $f : \Omega \to \mathbb{R}_0^+$ be a $(\star)$-integrable function. Then f is m-integrable and for every $E \in \Sigma$*

$$\int_E f dm = \int_E^\star f dm.$$

Proof: Let f be $(\star)$-integrable; then for every $E \in \Sigma$ there exists $w^E \in X$ such that for every $\varepsilon > 0$ there exists $\Delta(\varepsilon) : \mathbb{R}^+ \to \mathcal{A}$ which must be the same for every E, such that

$$\limsup_{n\to+\infty} \left\| w^E - \sum_{i\leq n} \phi^E(t_i)\lambda(T_i) \right\| \leq \varepsilon$$

for every generalized McShane partition $(T_i^\varepsilon, t_i)_{i\in\mathbb{N}}$ subordinate to $\Delta(\varepsilon)$.
Let $\varepsilon > 0$ be fixed. By Proposition 4.3 we can consider a generalized McShane partition subordinate to $\Delta(\varepsilon)$ of the form $([a_i^\varepsilon, a_{i+1}^\varepsilon], t_i^\varepsilon)_{i\in\mathbb{N}}$ such that $t_i^\varepsilon \in [a_i^\varepsilon, a_{i+1}^\varepsilon]$, $a_{i+1}^\varepsilon - a_i^\varepsilon < \varepsilon$ and $\bigcup_{i\in\mathbb{N}}[a_i^\varepsilon, a_{i+1}^\varepsilon] = \mathbb{R}_0^+$. Then, for every $E \in \Sigma$,

$$\limsup_{n\to+\infty} \left\| w^E - \sum_{i=1}^{n} \phi^E(t_i^\varepsilon)(a_{i+1}^\varepsilon - a_i^\varepsilon) \right\| \leq \varepsilon.$$

Now we want to show that it is possible to construct a simple function, indipendent of E, such that its Bochner integral is close to w^E.
For every $i \in \mathbb{N}$. we denote by $A_i^\varepsilon, C_i^\varepsilon$ the following sets:

$$A_i^\varepsilon = f^{-1}([a_i^\varepsilon, a_{i+1}^\varepsilon]), \qquad C_i^\varepsilon = f^{-1}(]t_i^\varepsilon, +\infty[)$$

Let $g_n^{(\varepsilon)} = \sum_{i=0}^{n}(a_{i+1}^\varepsilon - a_i^\varepsilon) \cdot 1_{C_i^\varepsilon}$. The simple function $g_n^{(\varepsilon)}$ is m and $(\star)$-integrable and

$$\int_E g_n^{(\varepsilon)} dm = \sum_{i=0}^{n}(a_{i+1}^\varepsilon - a_i^\varepsilon)\phi^E(t_i^\varepsilon).$$

Then, it follows that

$$\limsup_{n\to\infty} \left\| w^E - \int_E g_n^{(\varepsilon)} dm \right\| \leq \varepsilon.$$

Observe also that, if $\omega \in A_i^\varepsilon, i \leq n$

$$g_n^{(\varepsilon)}(\omega) = \begin{cases} a_i^\varepsilon & \text{if } a_i^\varepsilon \leq f(\omega) \leq t_i^\varepsilon \\ a_{n+1}^\varepsilon & \text{if } t_i^\varepsilon < f(\omega) \leq a_{n+1}^\varepsilon, \end{cases}$$

and therefore $|f(\omega) - g_n^{(\varepsilon)}(\omega)| \leq \varepsilon$ unifomly in $\bigcup_{i\leq n} A_i^\varepsilon$.

Now we want to show that there is a sequence $(g_n)_n$ of defining simple functions. Let $(\varepsilon_k)_k$ be a decreasing sequence of positive numbers converging to 0 and let $\sigma_k = 2\varepsilon_k$. Given ε_1 and the sequence $(g_n^{(\varepsilon_1)})_n$ there exists an integer $\overline{n}(\varepsilon_1)$ such that for every $n \geq \overline{n}(\varepsilon_1)$

$$\left\| w^E - \int_E g_n^{(\varepsilon_1)} dm \right\| < \sigma_1.$$

Then we set $b_1 = a^{(\varepsilon_1)}_{\overline{n}(\varepsilon_1)+1}$ and $g_1(\omega) = g^{(\varepsilon_1)}_{\overline{n}(\varepsilon_1)+1}(\omega)$. So

$$| x_0^* m | (| g_1 - f |> \sigma_1) \leq | x_0^* m | (f > b_1).$$

If $\omega \in A_i^{\varepsilon_1}$ with $i \leq \overline{n}(\varepsilon_1) + 1$ then

$$| g_1(\omega) - f(\omega) |< \varepsilon_1 < \sigma_1.$$

We consider now $\varepsilon_2 > 0$. Then there exists an integer $\widetilde{n}(\varepsilon_2) > \overline{n}(\varepsilon_1)$ such that for every $n \geq \widetilde{n}(\varepsilon_2)$

$$\left\| w^E - \int_E g_n^{(\varepsilon_2)} dm \right\| \leq \sigma_2.$$

We define now

$$b_2 = \min\{a_j^{(\varepsilon_2)} : a_j^{(\varepsilon_2)} \geq \max\{b_1 + 2, a_{\widetilde{n}(\varepsilon_2)+1}\}\}$$

Then we set $b_2 = a^{(\varepsilon_2)}_{\overline{n}(\varepsilon_2)+1}$ and $g_2 = g^{(\varepsilon_2)}_{\overline{n}(\varepsilon_2)+1}$. Thus

$$| x_0^* m | (| f - g_2 |> \sigma_2) \leq | x_0^* m | (f > b_2).$$

Iterating this procedure we obtain a sequence of integers $(n_k)_k$ where $n_k = \overline{n}(\varepsilon_k) + 1$, a sequence of real numbers $(b_k)_k$ such that $\lim_{k\to\infty} b_k = +\infty$ and a sequence of simple functions $(g_k)_k$ defined by $g_k = g_{n_k}^{(\varepsilon_k)}$ such that it fulfills the relationships

$$|x_0^* m|(|f - g_k| > \sigma_k) \leq |x_0^* m|(f > b_k)$$
$$\left\| \int_E g_k dm - w^E \right\| \leq \sigma_k.$$

So we have

$$\lim_{k\to\infty} \left\| \int_E g_k dm - w^E \right\| = 0.$$

It only remains to prove that g_k ν-converges to f, for any control ν.
Let $\alpha > 0$. Since $\lim_{k\to\infty} \sigma_k = 0$ there exists $\overline{k}$ such that for every $k \geq \overline{k}$, $\sigma_k < \alpha$.
Then

$$\{\omega \in \Omega : \mid g_k(\omega) - f(\omega) \mid > \alpha\} \subset \{\omega \in \Omega : \mid g_k(\omega) - f(\omega) \mid > \sigma_k\} \subset \{\omega \in \Omega : f(\omega) > b_k\}.$$

By Lemma 3.1 it follows

$$\lim_{k\to\infty} \nu(\omega : f(\omega) > b_k) = 0$$

and hence the convergence follows.

Before proving the converse implication, we point out that the results given in [4], section 2, hold also if X is not separable.

Proposition 4.5 *Let $f : \Omega \to \mathbb{R}_0^+$ be a bounded, measurable function. Then f is $(\star)$-integrable and the two integrals coincide.*

Proof: Since f is bounded let $I \subset \mathbb{R}_0^+$ be a bounded interval such that $f(x) \in I$ for every $x \in \Omega$. By using Lebesgue ladder trick it is possible to construct a sequence $(f_n)_n$ of simple functions which converges to f uniformly, with $f_n \leq f_{n+1} \leq f$ for every n.
We set now

$$h_n(t) \;=\; \|m\|(x \in \Omega : f(x) > t, f_n(x) \leq t); \qquad h_n^E(t) = \|m\|(x \in E : f(x) > t, f_n(x) \leq t).$$

Let $\varepsilon > 0$ be fixed and consider $\delta(\varepsilon) > 0$ such that if $\nu(A) \leq \delta$ then $\|m\|(A) \leq \varepsilon$.
By Theorem 3.2 of [4] ϕ is Bochner integrable, and let w_E be its integral. So first we want to prove that for every $\varepsilon > 0$ there exists n such that $\int_I \|\phi^E(t) - \phi_n^E(t)\| dt \leq \varepsilon$ for every $E \in \Sigma$.
We observe that the family $\{\|\phi^E(t) - \phi_n^E(t)\|, E \in \Sigma\}$ is such that for every $t \in \mathbb{R}_0^+$: $\Gamma(t) - \Gamma_n(t) \leq \delta$ hence $\|\phi^E(t) - \phi_n^E(t)\| \leq \varepsilon$ uniformly with respect to $E \in \Sigma$.
Since

$$\int_E f d\nu = \int_I \Gamma^E(t) dt, \qquad \lim_n \int_I \Gamma(t) - \Gamma_n(t) dt = 0$$

for every fixed $\varepsilon > 0$ there exists $\overline{n}$ such that for every $n \geq \overline{n}$

$$\int_I \Gamma(t) - \Gamma_n(t)dt \leq \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)} \cdot \delta\left(\frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\right).$$

Then, by Markov inequality,

$$\lambda\left(t \in \mathbb{R}_0^+ : \Gamma(t) - \Gamma_n(t) > \delta\left(\frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\right)\right) \leq \frac{\int_I \Gamma(t) - \Gamma_n(t)dt}{\delta\left(\frac{\varepsilon}{\|m\|(\Omega)+\lambda(I)}\right)} \leq \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}$$

and so, for every $E \in \Sigma$, by inclusion,

$$\lambda\left(t \in \mathbb{R}_0^+ : h_n^E(t) > \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\right) \leq \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}$$

in fact, if $t \in \mathbb{R}_0^+$ is such that

$$h_n^E(t) > \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}$$

then

$$\Gamma(t) - \Gamma_n(t) > \delta\left(\frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\right).$$

Then, for every $E \in \Sigma$, and for every $n \geq \overline{n}$

$$\begin{aligned}
\int_I \|\phi^E(t) - \phi_n^E(t)\|dt &\leq \int_I h_n^E(t)dt = \\
&= \int_{(t \in I: h_n^E(t) \leq \frac{\varepsilon}{\|m\|(\Omega)+\lambda(I)})} h_n^E(t)dt + \int_{(t \in I: h_n^E(t) > \frac{\varepsilon}{\|m\|(\Omega)+\lambda(I)})} h_n^E(t)dt \\
&\leq \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\lambda(I) + \|m\|(\Omega) \cdot \lambda\left(t : h_n^E(t) > \frac{\varepsilon}{\|m\|(\Omega) + \lambda(I)}\right) \leq \varepsilon.
\end{aligned}$$

We denote by $\Psi^E(t) = \phi_{\overline{n}}{}^E(t)$ and with w_0^E its Bochner integral. So

$$\|w^E - w_0^E\| \leq \varepsilon. \tag{1}$$

Since $f_{\overline{n}}$ is simple, then it is $(\star)$-integrable and so there exists a gauge Δ^0_ε such that for every $E \in \Sigma$

$$\limsup_{n\to\infty} \|w^E_0 - \sum_{i=1}^n \lambda(T_i)\Psi^E(t_i)\| \le \varepsilon \tag{2}$$

for every generalized Mc Shane partition $(T_i, t_i)_i$ subordinated to Δ^0_ε.

Now we want to prove that there exists a gauge Δ^1_ε such that for every $E \in \Sigma$ and for every generalized Mc Shane partition $(S_i, s_i)_i$ subordinated to Δ^1_ε

$$\sum_i \lambda(S_i)\|\phi^E(s_i) - \Psi^E(s_i)\| \le 2\varepsilon.$$

Consider $g(t) = h_{\overline{n}}(t)$, by Lemma 1J of [9], there exists a gauge Δ^1_ε such that

$$\sum_i \lambda(S_i)g(s_i) \le \int_I g(t)dt + \varepsilon$$

for every generalized Mc Shane partition $(S_i, s_i)_i$ subordinated to Δ^1_ε. So,

$$\begin{aligned} \sum_i \lambda(S_i)\cdot\|\phi^E(s_i) - \Psi^E(s_i)\| &= \sum_i \lambda(S_i)\cdot\|m(x\in E : f(x) > s_i, f_{\overline{n}}(x) \le s_i)\| \le \\ &\le \sum_i \lambda(S_i)\cdot {h_{\overline{n}}}^E(s_i) \le \sum_i \lambda(S_i)\cdot g(s_i) \le \\ &\le \int_I g(t)dt + \varepsilon \le 2\varepsilon. \end{aligned} \tag{3}$$

Let now $\Delta_\varepsilon = \Delta^1_\varepsilon \cap \Delta^0_\varepsilon$. Then for every generalized Mc Shane partition $(T_i, t_i)_i$ subordinated to Δ_ε, by (1), (2) and (3)

$$\begin{aligned} \limsup_n \quad & \|w^E - \sum_{i=1}^n \lambda(T_i)\phi^E(t_i)\| \le \|w^E - w^E_0\| + \limsup_n \|w^E_0 - \sum_{i=1}^n \lambda(T_i)\Psi^E(t_i)\| + \\ + \quad & \sum_i \lambda(T_i)\|\Psi^E(t_i) - \phi^E(t_i)\| \le 4\varepsilon. \end{aligned}$$

The equality between the two integrals follows by Theorem 4.4. .

Before proving the converse implication we need some preliminary technical lemmata.

In these lemmata if f is a function, set $f_n = f \wedge n$ and $\phi_n^{(\cdot)}(t) = m(x \in (\cdot) : f_n > t)$.

Lemma 4.6 *If $f : \Omega \to \mathbb{R}_0^+$ is m-integrable then, for every $B \in \mathcal{B}$ and for every $E \in \Sigma$*

$$\lim_{n\to\infty} \int_B \phi_n^E(t)dt \in X$$

and moreover, for every $x^\star \in X^\star$,

$$x^\star \left(\lim_{n\to\infty} \int_B \phi_n^E(t)dt \right) = \int_B x^\star \phi^E(t)dt.$$

Proof: For every $n \in \mathbb{N}$, let $\phi_n^E(t) = m\{\omega \in E : f(\omega) \wedge n > t\}$; then, by Proposition 4.5, ϕ_n^E is Mc Shane integrable and, for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$, there exists a gauge $\Delta_n(\varepsilon)$ which satisfies the definition of $(\star)$-integrability. Now we want to prove that $\lim_{n\to\infty} \int_B \phi_n^E(t)dt$ exists in X. Observe that for every $t \in \mathbb{R}_0^+, \phi_n^E(t)$ converges to $\phi^E(t)$ and, for every $B \in \mathcal{B}$,

$$\lim_{n\to\infty} \int_B \phi_n^E(t)dt \in X.$$

In fact, it suffices to prove that $\left(\int_B \phi_n^E(t)dt \right)_n$ is Cauchy in X for every $B \in \mathcal{B}$.
Let $x^\star \in X_1^\star$ be fixed, and let $n, p \in I\!N$ with $p > n$. Then

$$|< x^\star | \int_B \phi_n^E dt - \int_B \phi_p^E dt > | \leq 4 \sup_{A \subset E} \left\| \int_A (f_p - f_n) dm \right\|$$

which converges to zero since $(f_n)_n$ is Cauchy in $L^1(m)$.
Then for every $\varepsilon > 0$ there exists n_0 such that for every $n, p > n_0$ and for every $B \in \mathcal{B}$, $x^\star \in X_1^\star$

$$| < x^\star | \int_B \phi_n^E dt - \int_B \phi_p^E dt > | \leq \varepsilon,$$

and hence $\left(\int_B \phi_n^E dt \right)_n$ is Cauchy uniformly in E and B.

Since $f \in L^1(m)$ then, for every $x^\star \in X^\star, f \in L^1(x^\star m)$ and, by Lemma 3.5 of [4], $f \in L^1(|x^\star m|)$.
By Theorem 3.6 of [4] $f \in \widehat{L}^1(|x^\star m|)$, then for every $E \in \Sigma$

$$|x^\star \phi_n^E(t)| \leq |x^\star m|(f > t) \in L^1(\lambda)$$

and $x^\star \phi_n^E$ converges pointwise to $x^\star \phi^E$. So

$$x^\star \left(\lim_n \int_B \phi_n^E(t)dt \right) = \int_B x^\star \phi^E(t)dt.$$

Lemma 4.7 *If $f : \Omega \to \mathbb{R}_0^+$ is m-integrable then, for every convex combination of $\phi_i^{(\cdot)}$, $\phi_{co}^{(\cdot)} = \sum_{j\le n}\alpha_j \phi_j^{(\cdot)}$ and for every $\varepsilon > 0$ there exists a gauge Δ such that for every $E \in \Sigma$ and for every $k \in \mathbb{N}$,*

$$\left\| \int_B \phi_{co}^E(t)dt - \sum_{i\le k} \lambda(S_i)\phi_{co}^E(s_i) \right\| \le \varepsilon$$

for every partial Mc Shane partition $(S_i, s_i)_{i\in\mathbb{N}}$ of $\mathbb{R}_0^+$subordinated to Δ and such that $B = \bigcup_{i\le k} S_i$.

Proof: Let $n \in \mathbb{N}^+$, and let $(\alpha_0, \cdots, \alpha_n)$ be fixed in the $(n+1)$-th dimensional symplex. Let

$$\Gamma(t) = \sum_{j\le n} \alpha_j \cdot \nu(\{x \in \Omega : f(x) \wedge j > t\}),$$

and

$$\phi_{co}^E(t) = \sum_{j\le n} \alpha_j \cdot m(\{x \in E : f(x) \wedge j > t\}).$$

By construction Γ is a scalar Lebesgue integrable function and, by Lemma 2B of [9], for every $\sigma > 0$ there exists a gauge Δ_σ such that for every $k \in \mathbb{N}$

$$\left| \int_B \Gamma(t)dt - \sum_{i\le k} \lambda(S_i)\Gamma(s_i) \right| \le \sigma \tag{4}$$

for every generalized Mc Shane partition $(S_i, s_i)_{i\in\mathbb{N}}$ of $\mathbb{R}_0^+$subordinated to Δ_σ, where $B = \bigcup_{i\le k} S_i$. In fact a generalized Mc Shane partition $(S_i, s_i)_{i\in\mathbb{N}}$ is a partial one and we can apply Lemma 2B of [9] to $(S_i, s_i)_{i\le k}$.
Fix $\varepsilon > 0$. If we take

$$\sigma = \sigma(\rho) = \inf_{j\le n}\{\alpha_j : \alpha_j \ne 0\} \cdot \rho \cdot \tau(\rho)$$

where

$$\rho = \frac{\varepsilon}{4(n+1)[n + \|m\|(\Omega)]}$$

and $\tau(\cdot)$ is that of the absolute continuity of $\|m\|$ with respect to ν, we want to show that, for every $E \in \Sigma$,

$$\left\| \int_B \phi_{co}^E(t)dt - \sum_{i\leq k} \lambda(S_i)\phi_{co}^E(s_i) \right\| \leq \varepsilon. \tag{5}$$

Let $(S_i, s_i)_{i\in\mathbb{N}}$ be a partial Mc Shane partition subordinated to Δ_σ and let

$$V_i = S_i \cap [0, s_i[\qquad U_i = S_i \cap [s_i, +\infty[.$$

We can observe that the partitions

$$\begin{aligned} \Pi_1 &= \big\{(V_1, s_1), \cdots, (V_k, s_k), (U_1, s_1), \cdots (U_k, s_k), (S_{k+p}, s_{k+p}), p \in I\!N^+\big\} \\ \Pi_2 &= \big\{(U_1, s_1), \cdots (U_k, s_k), (V_1, s_1), \cdots, (V_k, s_k), (S_{k+p}, s_{k+p}), p \in I\!N^+\big\} \end{aligned}$$

are also subordinated to Δ_σ and so, by (4),

$$\left| \int_{\cup_{i\leq k} V_i} \Gamma(t)dt - \sum_{i\leq k} \lambda(V_i)\Gamma(s_i) \right| \leq \sigma; \tag{6}$$

$$\left| \int_{\cup_{i\leq k} U_i} \Gamma(t)dt - \sum_{i\leq k} \lambda(U_i)\Gamma(s_i) \right| \leq \sigma. \tag{7}$$

Set now

$$\Theta(t) = \Gamma(t) - \textstyle\sum_{i\leq k} 1_{S_i}(t) \cdot \Gamma(s_i).$$

If $t \in V_i, i \leq k$ then $t < s_i$ and

$$\Theta(t) = 1_{V_i}(t) \cdot \textstyle\sum_{j\leq n} \alpha_j \cdot \nu(\{x \in \Omega : t < f(x) \wedge j \leq s_i\})$$

while, if $t \in U_i, i \leq k$, $t \geq s_i$ and

$$\Theta(t) = -1_{U_i}(t) \cdot \textstyle\sum_{j\leq n} \alpha_j \cdot \nu(\{x \in \Omega : s_i < f(x) \wedge j \leq t\}).$$

Then, (6) and (7), become

$$\int_{\cup_{i\leq k}V_i}\Theta(t)dt \leq \sigma; \tag{8}$$

$$\int_{\cup_{i\leq k}U_i}-\Theta(t)dt \leq \sigma. \tag{9}$$

Let $E \in \Sigma$ be fixed and let

$$\psi_{co}^E(t) = \textstyle\sum_{i\leq k} 1_{S_i}(t)\cdot \phi_{co}^E(t).$$

For every $t \in B = \cup_{i\leq k} S_i$ we have

$$\begin{aligned}
& \phi_{co}^E(t) - \psi_{co}^E(t) = \left[\textstyle\sum_{i\leq k}1_{V_i}(t)\cdot\phi_{co}^E(t) - \sum_{i\leq k}1_{V_i}(t)\cdot\phi_{co}^E(s_i)\right] + \\
+ \quad & \left[\textstyle\sum_{i\leq k}1_{U_i}(t)\cdot\phi_{co}^E(t) - \sum_{i\leq k}1_{U_i}(t)\cdot\phi_{co}^E(s_i)\right] = \\
= \quad & \textstyle\sum_{i\leq k}1_{V_i}(t)\cdot\sum_{j\leq n}\alpha_j\cdot m(x\in E : t < f(x)\wedge j \leq s_i) + \\
- \quad & \textstyle\sum_{i\leq k}1_{U_i}(t)\cdot\sum_{j\leq n}\alpha_j\cdot m(x\in E : s_i \leq f(x)\wedge j < t);
\end{aligned}$$

and so, for every $t \in B$

$$\|\phi_{co}^E(t) - \psi_{co}^E(t)\| \leq \|m\|(\Omega)\textstyle\sum_{j\leq n}\alpha_j = \|m\|(\Omega).$$

Let now $\eta > 0$ be fixed. If $t \in V_i, i \leq k$ is such that

$$\|\phi_{co}^E(t) - \psi_{co}^E(t)\| > (n+1)\eta$$

then

$$\begin{aligned}
(n+1)\eta \quad &\leq \quad \left\|1_{V_i}(t)\textstyle\sum_{j\leq n}\alpha_j\ m(x\in E : t < f(x)\wedge j\leq s_i)\right\| \leq \\
&\leq \quad \textstyle\sum_{j\leq n}\alpha_j\ \|m\|(x\in E : t < f(x)\wedge j \leq s_i)
\end{aligned}$$

and so there exists $j^* \in \{0, 1, \cdots, n\}$ such that

$$\|m\|(x\in E : t < f(x)\wedge j^* \leq s_i) \geq \alpha_{j^*}\|m\|(x\in E : t < f(x)\wedge j^* \leq s_i) > \eta.$$

Since $\|m\| \ll \nu$ then it follows that

$$\nu(x \in \Omega : t < f(x) \wedge j^* \leq s_i) \geq \nu(x \in E : t < f(x) \wedge j^* \leq s_i) \geq \tau(\eta)$$

and so

$$\Theta(t) = \textstyle\sum_{j \leq n} \alpha_j \cdot \nu(x \in \Omega : t < f(x) \wedge j \leq s_i) \geq \alpha_{j^*} \cdot \tau(\eta).$$

Namely

$$\left\{t \in V_i : \|\phi^E_{co}(t) - \psi^E_{co}(t)\| > (n+1)\eta\right\} \subset \left\{t \in V_i : \Theta(t) > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\} \cdot \tau(\eta)\right\};$$

analogously

$$\left\{t \in U_i : \|\phi^E_{co}(t) - \psi^E_{co}(t)\| > (n+1)\eta\right\} \subset \left\{t \in U_i : -\Theta(t) > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\} \cdot \tau(\eta)\right\}$$

which means that

$$\left\{t \in S_i : \|\phi^E_{co}(t) - \psi^E_{co}(t)\| > (n+1)\eta\right\} \subset \left\{t \in S_i : |\Theta(t)| > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\} \cdot \tau(\eta)\right\}.$$

In particular, for $\eta = \rho$

$$\left\{t \in S_i : \|\phi^E_{co}(t) - \psi^E_{co}(t)\| > (n+1)\rho\right\} \subset \left\{t \in S_i : |\Theta(t)| > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\} \cdot \tau(\rho)\right\}.$$

From (8), (9) and by Markov inequality

$$\lambda(t \in S_i : |\Theta(t)| > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)) \leq \frac{1}{\inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)} \int_{S_i} |\Theta(t)| dt$$

and so

$$\begin{aligned}
& \lambda(t \in S_i : |\Theta(t)| > \inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)) \leq \\
\leq \quad & \frac{1}{\inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)} \int_{S_i} |\Theta(t)| dt \leq \frac{1}{\inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)} 2\sigma = \\
= \quad & \frac{1}{\inf_{j \leq n}\{\alpha_j : \alpha_j \neq 0\}\tau(\rho)} 2 \inf_{j \leq n}\{a_j : a_j \neq 0\}\rho\tau(\rho) = \\
= \quad & 2\rho = \frac{\varepsilon}{2(n+1)[n + \|m\|(\Omega)]}.
\end{aligned}$$

Then, by inclusion,

$$\lambda(t \in S_i : \|\phi^E_{co}(t) - \psi^E_{co}(t)\| > (n+1)\rho) \leq \rho.$$

So

$$\left\| \int_{B\cap[0,n]} \phi^E_{co}(t)dt - \textstyle\sum_{i\leq k} \lambda(S_i)\phi^E_{co}(s_i) \right\| = \left\| \int_{B\cap[0,n]} [\phi^E_{co}(t) - \psi^E_{co}(t)]dt \right\| \leq \tag{10}$$

$$\begin{aligned} &\leq \int_{B\cap[0,n]} \|\phi^E_{co}(t) - \psi^E_{co}(t)\|dt \leq \int_{\cup_{i\leq k} B\cap[0,n]\cap(t\in S_i : \|\phi^E_{co}(t)-\psi^E_{co}(t)\|>(n+1)\rho)} \|\phi^E_{co}(t) - \psi^E_{co}(t)\|dt + \\ &+ \int_{\cup_{i\leq k} B\cap[0,n]\cap(t\in S_i : \|\phi^E_{co}(t)-\psi^E_{co}(t)\|\leq(n+1)\rho)} \|\phi^E_{co}(t) - \psi^E_{co}(t)\|dt \leq \|m\|(\Omega)\cdot 2\rho + n(n+1)\rho \leq \\ &\leq 2\|m\|(\Omega)\cdot \frac{\varepsilon}{4(n+1)[n+\|m\|(\Omega)]} + n(n+1)\frac{\varepsilon}{4(n+1)[n+\|m\|(\Omega)]} \leq \varepsilon. \end{aligned}$$

Theorem 4.8 *If $f : \Omega \to \mathbb{R}^+_0$ is m-integrable, then f is $(\star)$-integrable, and for every $E \in \Sigma$*

$$\int_E f dm = \int_E^\star f dm.$$

Proof: Let $f_n = f \wedge n$; by Lemma 4.6, for every $B \in \mathcal{B}$ and for every $E \in \Sigma$, $\lim_n \int_B \phi^E_n(t)dt \in X$ so we can define $w^E_n, w^E : \mathcal{B} \to X$ as follows: $w^E_n(B) = \int_B \phi^E_n(t)dt, w^E(B) = \lim_{n\to\infty} w^E_n(B)$; moreover, for every $x^\star \in X^\star$,

$$x^\star\left(\lim_n \int_B \phi^E_n(t)dt\right) = \int_B x^\star\phi^E(t)dt. \tag{11}$$

Now we are going to construct a suitable family of sets so that a gauge similar to that in point (C) of Theorem 4A of [9] can be defined.
Let

$$\Gamma = \left\{(r, \alpha_0, \ldots, \alpha_n) : r, n \in \mathbb{N},\ \alpha_i \in Q \cap [0,1]\ \forall i = 0, \ldots, n,\ \ \sum_{i=0}^n \alpha_i = 1\right\}$$

For $\gamma \in \Gamma : \gamma = (r_\gamma, \alpha_0, \ldots, \alpha_n)$ let

$$\phi^E_\gamma = \sum_{i=0}^n \alpha_i \phi^E_i, \qquad \widehat{\phi}(t) = \|m\|(x \in \Omega : f(x) > t).$$

For every $\varepsilon > 0$ let $h : \mathbb{R}_0^+ \to \mathbb{R}^+$ be an integrable function such that $\int_0^\infty h(t)dt < \varepsilon$ and let

$$R_\gamma = \left\{ t \in [0, n[: h(t) > \frac{1}{1+r_\gamma}, \widehat{\phi}(t) \le r_\gamma, \sum_{j=0}^{n-1} 1_{[j,j+1[}(t) \sum_{i=0}^{j} \alpha_i \widehat{\phi}(t) \le h(t) \right\}.$$

It is $R_\gamma \in \mathcal{B}$ and $\mathbb{R}_0^+ = \bigcup_{\gamma \in \Gamma} R_\gamma$.

In fact, for every $t_0 \in \mathbb{R}_0^+$ fixed, $t_0 \in [[t_0], [t_0]+1[$ and, setting $\lambda_0 = \frac{\widehat{\phi}(t_0)}{h(t_0)}$, there exists $r_0 \in \mathbb{N}^+$ such that $h(t_0) > \frac{1}{1+r_0}$ and $\widehat{\phi}(t_0) \le r_0$, and there exists $q_0 \in Q \cap [1, \infty]$ such that $\lambda_0 < q_0$. If we set $n = [t_0]+1$ and $\gamma_0 = (r_0, 0, \cdots, 0, \frac{1}{q_0}, 1 - \frac{1}{q_0})$, then $t_0 \in R_{\gamma_0}$; in fact by construction

$$\sum_{j=0}^{[t_0]} 1_{[j,j+1[}(t_0) \sum_{i=0}^{j} \alpha_i \widehat{\phi}(t_0) = \alpha_{[t_0]} \widehat{\phi}(t_0) 1_{[\ [t_0],[t_0]+1[} = \frac{1}{q_0} \widehat{\phi}(t_0) 1_{[\ [t_0],[t_0]+1[} \le \frac{1}{\lambda_0} \widehat{\phi}(t_0) 1_{[\ [t_0],[t_0]+1[} = h(t_0)$$

and the other properties are easily verified.

Observe that, by construction, the level sets of f and f_n differ only for levels greater then or equal to n; therefore $\phi^E(t)$ and $\phi_n^E(t)$ are equal for $t < n$; since $\sum_{i=0}^n \alpha_i = 1$ by writing $\phi^E(t) = \sum_{i=0}^n \alpha_i \phi^E(t)$, we find that

$$\phi_\gamma^E(t) = \sum_{i=0}^{n} \alpha_i \phi_i^E(t) = \begin{cases} \sum_{i=1}^n \alpha_i \phi_i^E(t) & 0 \le t < 1, \\ \sum_{i=2}^n \alpha_i \phi_i^E(t) & 1 \le t < 2, \\ \vdots & \\ \alpha_n \phi_n^E(t) & n-1 \le t < n, \end{cases}$$

$$\phi^E(t) - \phi_\gamma^E(t) = \begin{cases} \alpha_0 \phi^E(t) & 0 \le t < 1, \\ \sum_{i=0}^1 \alpha_i \phi^E(t) & 1 \le t < 2, \\ \vdots & \\ \sum_{i=0}^{n-1} \alpha_i \phi^E(t) & n-1 \le t < n, \end{cases}$$

so, for every $E \in \Sigma$ and for every $t \in [0, n[$

$$\|\phi^E(t) - \phi_\gamma^E(t)\| \le \sum_{j=0}^{n-1} 1_{[j,j+1[}(t) \sum_{i=0}^{j} \alpha_i \widehat{\phi}(t) \tag{12}$$

Suppose now that $\gamma \in \Gamma$ and $H \in R_\gamma \cap \mathcal{B}$; then by (11),

$$\begin{aligned} \|w^E(H) - \int_H \phi^E_\gamma(t)dt\| &= \sup_{x^* \in X^*_1} \left| x^* w^E(H) - x^* \int_H \phi^E_\gamma(t)dt \right| \leq \sup_{x^* \in X^*_1} \int_H |x^*(\phi^E(t) - \phi^E_\gamma(t))|dt \leq \\ &\leq \int_H \|\phi^E(t) - \phi^E_\gamma(t)\|dt \leq \int_H h(t)dt \qquad (13) \end{aligned}$$

Let now $(R'_\gamma)_{\gamma\in\Gamma}$ be a disjoint family of measurable sets such that $\bigcup_\gamma R'_\gamma = \mathbb{R}^+_0$ and $R'_\gamma \subset R_\gamma$ for every γ.
Let $(\varepsilon_\gamma)_\gamma$ be a family of positive numbers such that $\sum_\gamma (1 + r_\gamma)\varepsilon_\gamma \leq \varepsilon$.
Let $\delta_\gamma = \frac{\varepsilon_\gamma}{\|m\|(\Omega)}$. For every $n \in \mathbb{N}$, and for every $B \in \mathcal{B}$ such that $\lambda(B) \leq \delta_\gamma$ we have $\| \int_B \phi^E_n(t)dt\| \leq \|m\|(\Omega)\lambda(B) \leq \varepsilon_\gamma$ for every $E \in \Sigma$. So $\|w^E(B)\| = \lim_n \|w^E_n(B)\| \leq \varepsilon_\gamma$.
Let G_γ be an open set which contains R_γ and such that $\lambda(G_\gamma - R_\gamma) \leq \min\{\varepsilon_\gamma, \delta_\gamma\}$.
For every $\gamma \in \Gamma$, by Lemma 4.7 applied to $\phi^{(\cdot)}_\gamma$ and ε_γ, there exists a gauge Δ_γ such that for every $E \in \Sigma$ and for k

$$\left\| \int_B \phi^E_\gamma(t)dt - \sum_{i\leq k} \lambda(T_i)\phi^E_\gamma(t_i) \right\| \leq \varepsilon_\gamma \qquad (14)$$

for every partial Mc Shane partition of $\mathbb{R}^+_0$subordinated to Δ_γ such that $B = \cup_{i\leq n} T_i$.
By 1J of [9] applied to h there exists a gauge Δ^* such that

$$\sum_{i\leq n} \lambda(T_i)h(t_i) \leq 2\varepsilon \qquad (15)$$

for every partial Mc Shane partition of $\mathbb{R}^+_0$subordinated to Δ^*.
For every $t \in R'_\gamma$ let

$$\Delta(t) = \Delta_\gamma(t) \cap G_\gamma \cap \Delta^*(t).$$

Δ is the suitable gauge to prove that f is $(\star)$-integrable.

As in part (d) of Theorem 4A of [9] one shows that

$$\limsup_n \|w^E(\mathbb{R}^+_0) - \sum_{i=1}^n \lambda(T_i)\phi^E(t_i)\| \leq 8\varepsilon,$$

for every $E \in \Sigma$ and for every generalized Mc Shane partition $(T_i, t_i)_i$ subordinated to Δ.
This proves that f is $(*)$-integrable. The equality between the two integrals follows from

Theorem 4.4.

Suppose now that X is separable. Then $\widehat{L}^1(m) \subset L^1(m) = L^{1,*}(m)$. In fact, by Proposition 3.6 of [4] the first inclusion holds and the equivalence between $L^1(m)$ and $L^{1,*}(m)$ is a consequence of Theorems 4.4, 4.8 above.

Moreover the example given in [4] shows that the first inclusion is proper.

To obtain the equivalence among the three integrations we have to introduce some suitable conditions on m.

Corollary 4.9 *If m admits a bounded Radon-Nikodym density with respect to ν, then*

$$\widehat{L}^1(m) = L^1(m) = L^{1,*}(m).$$

Proof: The first equivalence follows from Theorem 3.9 of [12].

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