A RADON-NIKODYM THEOREM FOR A PAIR OF BANACH-VALUED FINITELY ADDITIVE MEASURES(*)

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SOMMARIO.- *Facendo uso di una recente teoria dell'integrazione rispetto ad una misura finitamente additiva a valori in uno spazio di Banach (massa vettoriale) si ottiene un teorema di Radon-Nikodym per una coppia di masse vettoriali.*

SUMMARY.- *Using a recent integration theory with respect to a Banach- valued finitely additive measure, a Radon-Nikodym theorem is derived for a pair of Banach-valued measures.*

1. Introduction

One of the most interesting problems arising when dealing with finitely additive measures (f.a.m.'s) concerns the existence of a Radon-Nikodym derivative for a pair of f.a.m.'s λ, m, with $\lambda << m$. It is known that the classical Radon-Nikodym Theorem fails to be true in the finitely additive case unless some further assumption is fulfilled. The first result in this direction dates back to Maynard [10], where the case of two scalar f.a.m.'s defined on an *algebra* of sets is investigated. The scalar case has also been faced in Greco [7] for subadditive set functions using a De Giorgi-Letta integration theory ([4]), and more recently by Candeloro-Martellotti ([2]) for non-atomic f.a.m.'s

Besides the scalar case, the vector case has been recently studied: in [8] Hagood generalized Maynard's result to the case of a Banach-valued f.a.m. using the Dunford-Schwartz integration theory ([5]-chapter III), while in [3] the existence of a "weak" density has been established. Since an integration theory for a pair of Banach-valued f.a.m.'s has been most recently developed by Brooks-Martellotti [1] extending that of Dunford-Schwartz, it seemed natural to extend Maynard's result to this case. Actually in the theory of Stochastic Processes the existence of a scalar

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density for a pair of f.a.m.'s defined on an *algebra* and without non-atomicity assumptions could be particularly useful.

In this paper we extend the Maynard's result for a pair of Banach-valued f.a.m.'s: to do this we make use of a condition which is equivalent to that assumed in Maynard [10] and in Hagood [8], but that turns out to be strictly stronger in the case here considered, as shown by means of an example.

2. Preliminaires

Let Ω be an abstract set, Σ be an algebra on Ω. Throughout this paper X will denote a separable Banach space, with dual X^*, $m : \Sigma \to X$ will denote a finitely additive strongly bounded measure. The following result holds

THEOREM 2.1. (Kats [9]) *The range $R(m)$ is bounded.*

From this theorem, we can define the *semivariation* of m, as the positive subadditive set function on Σ defined by

$$|m|(E) = \sup\{\|m(A)\|, A \subseteq E, A \in \Sigma\}$$

and we will denote by $|m|^*$ the extended semivariation according to [5].

DEFINITION 2.2. *A finitely additive measure λ is m- continuous, written $\lambda \ll m$, if for each $\epsilon > 0$ there exists $\sigma > 0$ such that $E \in \Sigma$ and $|m|(E) < \epsilon$ implies $|\lambda|(E) < \sigma$.*

We will say that a positive finitely additive measure $\nu : \Sigma \to \boldsymbol{R}_0^+$ is a *control* for m if ν and $|m|$ are equivalent. If ν is a control for m, then ν is said to be a *Rybakov control* if $\nu = |x^* m|$ for some $x^* \in X^*$.

The following theorem is known:

THEOREM 2.3. (Rybakov [11]) *If $m : \Sigma \to X$ is strongly bounded, then it admits a Rybakov control.*

m-null sets, m-null functions and m-simple functions will be defined as in [5].

DEFINITION 2.4. *A function $f : \Omega \to \boldsymbol{R}$ will be said to be "totally measurable" provided f is the limit in m-measure of a sequence of m-simple functions.*

In [1] a theory of integration is developed for the below defined integral

DEFINITION 2.5. (Brooks-Martellotti [1]) *A measurable function* $f: \Omega \to \boldsymbol{R}_0^+$ *will be said to be "m-integrable" if there exists a sequence* $(f_n)_n$ *of simple functions such that* $(f_n)_n$ *converges in v-measure to f, and the sequence* $(\int_F f_n dm)_n$ *converges in X uniformly with respect to F in* Σ. *In this case the m-integral of f is defined by*

$$\int_{(\cdot)} f\,dm = \lim_{n\to\infty} \int_{(\cdot)} f_n\,dm\ .$$

In ([1]) the above defined integral is compared with the vector extension of the following integral with respect to a non negative subadditive monotone set function.

DEFINITION 2.6. (De Giorgi-Letta [4]) *Let* $\sigma: \Sigma \to \boldsymbol{R}_0^+$ *be a set function such that*

i) $\sigma(\emptyset) = 0$;
ii) *if* $A, B \in \Sigma$, *and* $A \subseteq B$ *then* $\sigma(A) \leq \sigma(B)$;
iii) *if* $A, B \in \Sigma$ *and* $A \cap B = \emptyset$ *then* $\sigma(A \cup B) \leq \sigma(A) + \sigma(B)$.

For a Σ*-measurable function* $f: \Omega \to \boldsymbol{R}_0^+$ *we set*

$$(2.6.1)\quad \int_\Omega f\,d\sigma = \int_0^{+\infty} \sigma(f > t)dt;$$

while if $f: \Omega \to \boldsymbol{R}$, *we set:*

$$\int_\Omega f\,d\sigma = \int_\Omega f^+ d\sigma - \int_\Omega f^-\,d\sigma.$$

Let Σ^+ be the subset of Σ consisting of the sets with positive semivariation and let Σ^2 be the subset of Σ consisting of the sets E such that $|m|(E) < 2\|m(E)\|$.

DEFINITION 2.7. *We say that a countable (or finite) disjoint collection* $\mathcal{C}$ *in* Σ^+ *is "m-exhausting" in* Ω *(or, equivalently, that* $\mathcal{C}$ *exhausts m) if given any* $\epsilon > 0$ *there exists a finite subset I of N such that* $|m|(\Omega - \cup_{i\in I} E_i) < \epsilon$

If $\mathcal{C}$ is m-exhausting, and if every element of $\mathcal{C}$ satisfies a property P fixed, then the collection $\mathcal{C}$ is a P-*exhaustion* (or, equivalently, we say that P exhausts m).

DEFINITION 2.8. *A set property P is "null difference" if whenever* $E, F \in \Sigma^+$ *and* $|m|(E \Delta F) = 0$ *then either E, F satisfy P or neither.*

From this definition it follows that if (Ω, Σ, m) is a complete measure space and P is a null difference m- exhaustive property then it is possible to obtain a P-exhaustion $\{E_i\}_{i \in I}$ such that $\Omega = \cup_i E_i$.

For any $E \in \Omega$ and $\mathcal{C} \subset \Sigma$, let $E\mathcal{C} = \{A \in \mathcal{C} : A \subseteq E\}$. For fixed $E \in \Sigma^+$ and $\epsilon > 0$, we denote by $A(E\mathcal{C})$ the *average range* of λ with respect to m on E defined by:

$$A(E\mathcal{C}) = \left\{ \frac{\|\lambda(F)\|}{\|m(F)\|} : F \subseteq E,\ F \in \mathcal{C},\ \|m(F)\| \neq 0 \right\},$$

and by $A(E, \epsilon)$ the ϵ-*approximate average range* defined by:

$$A(E, \epsilon) = \{x \in \boldsymbol{R} : \|\lambda(F) - xm(F)\| < \epsilon |m|(F), \forall F \subset E, F \in \Sigma\}.$$

In addition we will use the notation $\delta(E)$ to denote the diameter of a set $E \subset \Omega$.

We state now two lemmata which will be used in the following.

LEMMA 2.9. *If* $m : \Sigma \to X$ *is a strongly bounded finitely additive Banach-valued measure and* $E \in \Sigma^+$, *then either* $E \in \Sigma^2$ *or there exists* $F \subset E$ *such that* $F \in \Sigma^2$.

Proof. Suppose not. Then for every $F \subset E$, $F \in \Sigma$ we have $|m|(F) \geq 2\|m(F)\|$; but by definition $|m|(E) = \sup \|m(H)\|$ where the supremum is taken over all the Σ-measurable subsets of E. Therefore it follows that

$$|m|(E) = \sup \|m(H)\| \leq \frac{1}{2}|m|(H) \leq \frac{1}{2}|m|(E).$$

Contradiction.

LEMMA 2.10. *Let f be a bounded measurable function, then:*

$$\| \int_E f\, dm \| \leq \int_E |f|\, d|m| \text{ for every } E \in \Sigma$$

where the right hand side is defined as in Definition 2.6.

Proof. The m-integrability is a consequence of [1] (Theorem 5). By means of the same result we have for non-negative f

$$\| \int_E f\, dm \| = \| \int_0^{+\infty} m(f 1_E > t) dt \| \leq \int_0^{+\infty} \| m(f 1_E > t) \| dt \leq$$

$$\leq \int_0^{+\infty} |m|(f 1_E > t) dt = \int_E f\, d|m|$$

whence, for arbitrary f, by making use of the decomposition

$$\| \int_E f\, dm \| = \| \int_E (f^+ - f^-) dm \| \leq \| \int_E f^+ dm \| + \| \int_E f^- dm \|$$

the assertion follows.

3. The Radon-Nikodym theorem

We begin with some preliminary result.

LEMMA 3.1. *Let m and λ be two finitely additive Banach-valued measures, with m strongly bounded, satisfying:*

i) $\lambda << m$;
ii) $A(\Omega\Sigma^2)$ *is bounded*;
iii) *for each* $\epsilon > 0$, *the set property* $A(E, \epsilon) \neq \emptyset$ *exhausts* m *on each element of* Σ^+.

Then there exists an m-integrable function f such that $\lambda(E) = \int_E f\, dm$ for all $E \in \Sigma$.

Proof. We may assume that (Ω, Σ, m) is complete since a function integrable with respect to the completion is integrable with respect to the initial space and has the same integral values.

As the property $A(E, \epsilon) \neq \emptyset$ is null difference, for every $\epsilon > 0$ and $E \in \Sigma^+$ there exists an exhaustion $(E_i)_i$ of m on E such that $E = \cup_i E_i$ and $A(E_i, \epsilon) \neq \emptyset$ for every i.

Let $(E_i^1)_i$ be an exhaustion of m such that $\Omega = \cup_i E_i^1$ and $A(E_i^1, 2^{-1}) \neq \emptyset$ for all i. We may decompose each E_i^1 in an exhausting way and by inductivity we may construct a sequence of exhaustions satisfying:

(3.1.1.) $A(E_\alpha^n, 2^{-n}) \neq \emptyset$ for all n and $\alpha \in N^n$;

(3.1.2.) $E_\alpha^n = \cup_i E_{\alpha,i}^{n+1}$ and $(E_{\alpha,i}^{n+1})$ exhausts m on E_α^n for every $n \in N$ and $\alpha \in N^n$;

(3.1.3.) $\Omega = \cup_\alpha E_\alpha^n$ and $(E_\alpha^n)_\alpha$ exhausts m for n fixed and α ranging on N^n.

Let now $f_n : \Omega \to R$ be defined by: $f_n = \Sigma_\alpha r_\alpha^n 1_{E_\alpha^n}$ where $r_\alpha^n \in A(E_\alpha^n, 2^{-n})$.

Then the functions f_n fullfill the following properties:

(3.1.4.) $(f_n)_n$ *is uniformly bounded.*

Proof. For any n and α by Lemma 2.9. there exists a set $F \in \Sigma^2$, $F \subset E_\alpha^n$. Then, denoted by M the supremum of $A(\Omega\Sigma^2)$,

$$|r_\alpha^n| = \left| r_\alpha^n + \frac{\|\lambda(F)\|}{\|m(F)\|} - \frac{\|\lambda(F)\|}{\|m(F)\|} \right| \leq \left| r_\alpha^n + \frac{\|\lambda(F)\|}{\|m(F)\|} \right| + \frac{\|\lambda(F)\|}{\|m(F)\|} \leq$$

$$\leq \frac{1}{\|m(F)\|} \|\lambda(F) - r_\alpha^n m(F)\| + \frac{\|\lambda(F)\|}{\|m(F)\|}$$

but $\dfrac{|m|(F)}{\|m(F)\|} < 2$ and hence

$$|r_\alpha^n| \leq 2^{-n} \frac{|m|(F)}{\|m(F)\|} + \frac{\|\lambda(F)\|}{\|m(F)\|} \leq 2^{1-n} \frac{\|\lambda(F)\|}{\|m(F)\|} \leq 1 + M$$

(3.1.5.) *The* f_n*'s are totally measurable and* m- *integrable.*

Since $(E_\alpha^n)_\alpha$ is exhausting in Ω, the finite sums $(\Sigma_{\alpha<(p,\ldots,p)} r_\alpha^n 1_{E_\alpha^n})$ with $p \in N$ converge in m-measure to f_n for $p \to \infty$ and hence f_n is totally measurable and m- integrable.

(3.1.6.) *The sequence $(f_n)_n$ is uniformly Cauchy.*

Fix k and $\alpha \in N^n$; for each fixed $n > k$ and $\beta \in N^{n-k}$, $(E^n_{\alpha\beta})$ is a decomposition of E^k_α. Then

$$|f_k(s) - f_n(s)| = |\sum_\alpha r^k_\alpha 1_{E^k_\alpha} - \sum_{\alpha\beta} r^n_{\alpha\beta} 1_{E^n_{\alpha\beta}}| =$$

$$\left|\sum_\alpha \sum_\beta r^k_\alpha 1_{E^k_{\alpha\beta}} - \sum_{\alpha\beta} r^n_{\alpha\beta} 1_{E^n_{\alpha\beta}}\right| = \left|\sum_{\alpha\beta} (r^k_\alpha - r^n_{\alpha\beta}) 1_{E^n_{\alpha\beta}}\right|$$

and since $r^k_\alpha, r^n_{\alpha\beta} \in A(E^n_{\alpha\beta}, 2^{-k})$, it follows that $|f_k(s) - f_n(s)| \leq 2^{1-k}$ holds for every $s \in \Omega$. Hence f_n is uniformly Cauchy.

By (3.1.4.), (3.1.5.) and (3.1.6.) the function $f = \lim_{n\to\infty} f_n$ is bounded and m-integrable. Indeed the uniform convergence implies the $|m|$-convergence and, a fortiori, the convergence in ν-measure where ν is any control measure. Moreover $\int_{(\cdot)} f_n dm << \nu(\cdot)$ uniformly with respect to n. In fact for $n \in N$ fixed one has:

$$\|\int_{(\cdot)} f_n\, dm\| = \|\int_{(\cdot)} (f_n^+ - f_n^-) dm\| = \|\int_{(\cdot)} f_n^+\, dm - \int_{(\cdot)} f_n^-\, dm\| \leq$$

$$\leq \|\int_{(\cdot)} f_n^+\, dm\| + \|\int_{(\cdot)} f_n^-\, dm\|.$$

By Lemma 2.10.

$$\|\int_{(\cdot)} f_n^\pm\, dm\| \leq \int_{(\cdot)} f_n^\pm\, d|m|;$$

by (3.1.4.) one has

$$\|\int_{(\cdot)} f_n^\pm dm\| \leq M|m|(\cdot),$$

and so $\|\int_{(\cdot)} f_n dm\| \leq 2M|m|(\cdot)$ uniformly with respect to n. Then by the Vitali Convergence Theorem [1] f is m-integrable and

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f\, dm \qquad \text{for every } E \in \Sigma.$$

By means of ([1] Theorem 5) one has:

$$\int_E f_n\,dm = \int_\Omega f_n\,1_E\,dm = \int_\Omega \left(\sum_\alpha r_\alpha^n\,1_{E\cap E_\alpha^n}\right) dm$$

$$\lim_{p\to\infty} \int_\Omega \left(\sum_{\alpha<(p,\dots p)} r_\alpha^n\,1_{E\cap E_\alpha^n}\right) dm = \lim_{p\to\infty}\left[\sum_{\alpha<(p,\dots p)} r_\alpha^n\,1_{E\cap E_\alpha^n}\right] =$$

$$= \sum_\alpha r_\alpha^n\, m(E\cap E_\alpha^n) = \sum_\alpha \int_{E\cap E_\alpha^n} f_n\,dm\,.$$

This shows that for $\epsilon > 0$ fixed one can choose $p \in N$ such that

$$\Big\| \sum_{\alpha<(p,\dots p)} r_\alpha^n\, m(E\cap E_\alpha^n) - \int_E f_n\,dm \Big\| < \frac{\epsilon}{2}\,.$$

Moreover, since $(E_\alpha^n)_\alpha$ exhausts E, p can be chosen in such a way that

$$|m|\Big[E - \bigcup_{\alpha<(p,\dots p)} (E\cap E_\alpha^n)\Big] < \sigma\Big(\frac{\epsilon}{2}\Big)$$

with σ as in Definition 2.2. Therefore we find:

$$\|\lambda(E) - \int_E f_n dm\| \le \|\lambda(E) - \lambda\Big[\bigcup_{\alpha<(p,\dots p)} (E\cap E_\alpha^n)\Big]\| +$$

$$+ \|\lambda\Big[\bigcup_{\alpha<(p,\dots p)} (E\cap E_\alpha^n) - \sum_{\alpha<(p,\dots p)} r_\alpha^n\, m(E\cap E_\alpha^n)\| +$$

$$+ \Big\| \sum_{\alpha<(p,\dots p)} r_\alpha^n\, m(E\cap E_\alpha^n) - \int_E f_n dm \Big\| \le$$

$$\le \|\lambda[E - \bigcup_{\alpha<(p,\dots p)} (E\cap E_\alpha^n)]\| + \Big\| \sum_{\alpha<(p,\dots p)} [\lambda(E\cap E_\alpha^n) - r_\alpha^n m(E\cap E_\alpha^n)]\Big\| + \frac{\epsilon}{2} \le$$

$$\le \frac{\epsilon}{2} + \sum_{\alpha<(p,\dots p)} 2^{-n}\,|m|(E\cap E_\alpha^n) + \frac{\epsilon}{2} \le \epsilon + 2^{-n}\,|m|(E)\,.$$

By the arbitrariness of $\epsilon > 0$, we have proven that

$$\|\lambda(E) - \int_E f_n dm\| \leq 2^{-n} |m|(E)$$

whence $\lambda(E) = \lim_{n\to\infty} \int_E f_n dm = \int_E f\, dm$ for every $E \in \Sigma$.

LEMMA 3.2. (EXHAUSTION PRINCIPLE) *Let $m : \Sigma \to X$ be a finitely additive measure. Then the following statements are equivalent:*

(3.2.1.) *for every $\epsilon > 0$ the set property $A(E, \epsilon) \neq \emptyset$ exhausts m on each element of Σ^+;*

(3.2.2.) *for every $\delta > 0$ there exists $C \in \Sigma$ and $\alpha \in (0, 1)$ such that:*

i) $|m|(\Omega - C) < \delta$;

ii) $\forall E \in C\Sigma^+$ *there exists* $F \in E\Sigma^+$ *such that* $|m|(F) > \alpha |m|(E)$ *and* $A(F, \epsilon) \neq \emptyset$.

The proof is exactly the same as that of Proposition 3.2. of [8] if one specifies property P as in (3.2.1.). Indeed it is

$$|m|(E) \leq \sum_{k=1}^{n} |m|(E \cap E_k) \leq \sum_{k=1}^{n} \frac{1}{2n} |m|(E) = \frac{1}{2} |m|(E) < |m|(E).)$$

THEOREM 3.3. (RADON-NIKODYM THEOREM) *Let m and λ be two finitely additive Banach-valued measures, with m strongly bounded. Then the following statements are equivalent:*

(3.3.1.) *there exists an m-integrable function f such that $\lambda(E) = \int_E f\, dm$ for every $E \in \Sigma$;*

(3.3.2.)

a) $\lambda << m$;

b) *for every $\epsilon, \delta > 0$, there exists $C \in \Sigma^+$ and $\alpha \in (0, 1)$ satisfying:*

i) $|m|(\Omega - C) < \delta$;

ii) $A(C\Sigma^2)$ *is bounded*;

iii) $\forall E \in C\Sigma^+ \; \exists\, F \in E\Sigma^+$ *such that* $|m|(F) > \alpha |m|(E)$ *and* $A(F, \epsilon) \neq \emptyset$.

Proof.

"If" part. Suppose $\lambda(E) = \int_E f\,dm$. Then a) is known ([1]). Let $\epsilon, \delta > 0$; then there exists a sequence $(f_n)_n$ which ν-converges to f. Since ν is equivalent to $|m|$, f_n $|m|$- converges to f.

Let $g = \Sigma_n x_n X_{E_n}$, where $\{E_n\}_n$ is a finite decomposition of Ω such that, for every n, $|m|^*\{s \in R : |f(s) - g(s)| > \frac{\epsilon}{2}\} < \delta$, and let $A \in \Sigma$ be a set such that $|m|(A) < \delta$ and $A \supset \{s \in R : |f(s) - g(s)| > \frac{\epsilon}{2}\}$; the set $C = \Omega - A$ satisfies i).

We shall prove now ii): take $M = \sup |x_n| + \frac{\epsilon}{2}$; then for all $s \in C$, $|f(s)| < M$. For every fixed $E \in C\Sigma^2$ we have:

$$\|\lambda(E)\| = \|\int_E f^+ dm - \int_E f^- dm\| \leq \|\int_E f^+ dm\| + \|\int_E f^- dm\|$$

and since $f^+ \leq 2M, f^- \leq 2M$ by Lemma 2.10

$$\|\int_E f^{\pm} dm\| = \|\int_0^{+\infty} m(\{\omega \in \Omega : f^{\pm}(\omega) > t\} \cap E)dt\| \leq$$

$$\leq \int_0^{+\infty} \|m(\{\omega \in \Omega : f^{\pm}(\omega) > t\} \cap E)\|\,dt \leq$$

$$\leq \int_0^{+\infty} |m|(\{\omega \in \Omega : f^{\pm}(\omega) > t\} \cap E)dt =$$

$$= \int_E f^{\pm} d|m| \leq 2M|m|(E).$$

From this it follows that

$$\frac{\|\lambda(E)\|}{\|m(E)\|} \leq \frac{4M|m|(E)}{\|m(E)\|} \leq 8M$$

which proves the boundedness of $A(C\Sigma^2)$.

To prove iii) let $\alpha = \frac{1}{2n}$ and $E \in C\Sigma^+$ be given. Then $|m|(E) \leq \Sigma_n |m|(E \cap E_n)$ and there exists $j \in N$ such that $|m|(E \cap E_j) > \alpha |m|(E)$

(otherwise $|m|(E) \leq \Sigma_{k=1}^{n} |m|(E \cap E_k) \leq \Sigma_{k=1}^{n} \frac{1}{2n}|m|(E) = \frac{1}{2}|m|(E) < |m|(E)$).

Take $F = E \cap E_j$; we shall prove that F satisfies iii). In fact for every $B \in F\Sigma$ we have:

$$\|\lambda(B) - x_j m(B)\| = \|\lambda(B) - \int_B g\, dm\| = \|\int_B (f - g) dm\| =$$

$$\|\int_B (f - g)^+ dm - \int_B (f - g)^- dm\| \leq \|\int_B (f - g)^+ dm\| + \|\int_B (f - g)^- dm\|$$

and since $\|\int_B (f - g)^{\pm} dm\| \leq \frac{\epsilon}{2}|m|(B)$ we have

$$\|\lambda(B) - x_j m(B)\| \leq \epsilon |m|(B)$$

and hence $x_j \in A(F, \epsilon)$.

"Only if" part. By Lemma 3.2. conditions i) and ii) are equivalent to condition iii) of Lemma 3.1. and hence all the assumption of Lemma 3.1. are verified. This concludes the proof.

In Maynard ([10] Lemma 3.7.) and in Hagood ([8] Lemma 3.4.) the assumption 3.3.2. b iii) is equivalent to the following one:

(*) for every $E \in \Sigma^+$ there exists $F \in E\Sigma^+$ such that $|m|(F) > \alpha |m|(E)$ and $\delta(A(F\Sigma^2)) < \epsilon$.

This equivalence fails to be true in the case here examined, and we shall show that Theorem 3.3. cannot be proved under this weaker condition. Indeed we shall exhibit two finitely additive measures, λ, m satisfying (*) and the other assumptions of Theorem 3.3., but not 3.3.2. b iii); obviously, since Theorem 3.3. is a necessary and sufficient condition, it will turn out that a Radon-Nikodym derivative cannot exist, i.e. 3.3.1. is not fulfilled.

EXAMPLE 3.4.

Let Ω be an abstract set and let $\Sigma = P(\Omega)$. Let $\mu : \Sigma \to [0, 1]$ be a strongly bounded finitely additive probability measure. Let λ and m be the finitely additive measures defined by: $m = (\mu, \mu)$ and $\lambda = (-\mu, \mu)$.

Then, λ is absolutely continuous with respect to m and, for every $\epsilon, \delta > 0$ setting $C = \Omega$, we find:

- $|m|(\Omega - C) < \delta$;

- $A(\Omega\Sigma^2) = \left\{ \frac{\|\lambda(F)\|}{\|m(F)\|},\ F \in \Sigma^+,\ \|m(F)\| \neq 0 \right\} = \{1\}$

i.e. assumptions 3.3.2. b i)-ii) are fulfilled.

Now, since $\|m(\cdot)\| = \sqrt{2}\,\mu(\cdot)$, we have $|m|(\cdot) = \|m(\cdot)\|$, whence

- for every $E \in \Sigma^+$ there exist $F \in E\Sigma^+$ and $\alpha \in (0, 1)$ such that $|m|(F) > \alpha \sup_{F \subset E} \|m(F)\| = \alpha |m|(E)$ and $\delta(A(F\Sigma^2)) < \epsilon$ i.e. (*) holds.

We now want to prove that for every $E \in \Sigma^+$ and $F \in E\Sigma^+$ there exists $\epsilon > 0$ such that $\delta(A(F\Sigma^2)) < \epsilon$ but $A(F, \epsilon) = \emptyset$.

To prove this it suffices to show that for every $x \in \boldsymbol{R}$ and for every $H \in F\Sigma^+$ one has:

$$\left\| \frac{\lambda(H)}{\|m(H)\|} - \frac{xm(H)}{\|m(H)\|} \right\| \geq 2\epsilon .$$

Indeed, since $\Sigma^2 = \Sigma^+$, and $\Sigma^2 \neq \emptyset$, we have:

$$\frac{\|\lambda(H) - xm(H)\|}{|m|(H)} > \frac{\|\lambda(H) - xm(H)\|}{2\|m(H)\|} = \frac{1}{2} \left\| \frac{\lambda(H)}{\|m(H)\|} - \frac{xm(H)}{\|m(H)\|} \right\| .$$

Now, for $\epsilon = \frac{1}{2}$, it is:

$$\left\| \frac{\lambda(H)}{\|m(H)\|} - \frac{xm(H)}{\|m(H)\|} \right\| = \left\| \frac{(\mu(H) - x\mu(H); -\mu(H) - x\mu(H))}{\sqrt{2\mu^2(H)}} \right\|$$

$$= \sqrt{\frac{(1-x)^2\mu^2(H) + (1+x)^2\mu^2(H)}{2\mu^2(H)}} = \sqrt{1+x^2} \quad \geq 1$$

from which the conclusion follows.

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