# Abstract Integration in Convergence Groups *

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SUNTO. Si introduce una definizione di integrale rispetto a funzioni d'insieme finitamente additive a valori in opportuni "gruppi di convergenza". Si dimostrano inoltre teoremi tipo-Vitali.

SUMMARY. A definition of integral is given, with respect to finitely additive set functions, with values in suitable groups, endowed with a structure of "convergence", defined axiomatically. Furthermore, some Vitali-type convergences are proved.

## 1 Introduction.

This paper is a natural sequel of [3]. Here, we consider abstract structures, similar to the so-called "convergence-groups", which were introduced by Fischer ([7]) and were investigated by several authors (a rich survey of such types of studies is found in [16]; see also [5], [8], [15], [18]). Other types of "abstract" spaces were investigated also by Avallone and Basile ([1]), Kusraev and Malyugin ([9]), Nakanishi ([14]) and others.

In this paper an integral is defined, for $R_1$-valued functions, defined on an arbitrary set $X$, with respect to a $R_2$-valued (finitely additive) mean $\mu : \Sigma \to R_2$, where $\Sigma \subset \mathcal{P}(X)$ is an algebra, and $R_1$ and $R_2$ are convergence groups, "linked together" by some kind of bilinear mappings. Our integral will be an element of another convergence group $R$.

The results here obtained extend both the cases of Riesz spaces (see [3]) and of topological groups (see [11], [12] and [17]).

Finally, we give a comparison with some classical integrals, like Bochner, Pettis and stochastic integral.

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## 2 Preliminaries.

Throughout this section, we assume that $Z$ is an arbitrary set, $R$ is an Abelian group, $S \equiv R^{I\!N}$ is the set of all sequences in $R$.

**Definition 2.1** We say that $R$ is a *pre-convergence group* if there exists a subgroup $T \subset S$, satisfying the following properties:

1. If $(r_n)_n$ is a definitively constant sequence (i.e. there exist $r \in R$ and $n_0 \in I\!N$, such that $r_n = r, \ \forall\ n \in I\!N, \ n \geq n_0,$) then $(r_n)_n$ belongs to $T$;
2. if $(r_n)_n \in T$, then every subsequence of $(r_n)_n$ belongs to $T$;

   and there exists an additive map $l : T \to R$, such that
3. If $(r_n)_n$ is definitively constant, $r_n = r \ \ \forall\ n \geq n_0$, then $l((r_n)_n) = r$.
4. If $l((r_n)_n) = r$, and $(r_{n_k})_k$ is a subsequence of $(r_n)_n$, then $l((r_{n_k})_k) = r$.

The element $\lambda \equiv l((r_n)_n)$ will be called *limit* of the involved sequence, and we will write $\lim_n \ r_n = \lambda$.

A consequence of 1.-4. of Definition 2.1 is that, for each sequence $(r_n)_n \in T$ and $\lambda \in R$, one has:

$$[\lim_n r_n = \lambda] \Longleftrightarrow [\lim_n (r_n - \lambda) = 0.]$$

**Definition 2.2** Let $\mathcal{S} \equiv \{(\psi_n)_n : \ \ \psi_n : Z \to R\}$. A pre-convergence group $R$ is said to be a $(Z)$-*convergence group* (or *convergence group* ) if: there exist a subgroup $\mathcal{T}_Z \subset \mathcal{S}$, and a map $\Phi : \mathcal{T}_Z \to R^Z$, satisfying the following properties:

1. If $(\psi_n)_n$ is a definitively constant sequence of functions (i.e. there exist $\psi \in R^Z$ and $n_0 \in I\!N$, such that $\psi_n = \psi, \ \forall\ n \in I\!N, \ n \geq n_0$) then $(\psi_n)_n$ belongs to $\mathcal{T}_Z$;
2. If $(\psi_n)_n \in \mathcal{T}_Z$, then every subsequence of $(\psi_n)_n$ belongs to $\mathcal{T}_Z$;

   there exists an additive map $\Phi : \mathcal{T}_Z \to R$, such that
3. If $(\psi_n)_n$ is definitively constant, $\psi_n = \psi \ \ \forall\ n \geq n_0$, then $\Phi((\psi_n)_n) = \psi$.
4. If $\Phi((\psi_n)_n) = \psi$, and $(\psi_{n_k})_k$ is a subsequence of $(\psi_n)_n$, then $\Phi((\psi_{n_k})_k) = \psi$.
5. For each sequence $(z_n)_n$ in $Z$ and every $(\psi_n)_n \in \mathcal{T}_Z, \ \lim_n \ [\psi_n(z_n) - \psi(z_n)] = 0$.

The element $\psi \equiv \Phi((\psi_n)_n)$ will be called $(Z)$-*limit* of the involved sequence. We will write $(Z) - \lim_n \ \psi_n = \psi$, or $(Z) - \lim_n \ \psi_n(z) = \psi(z)$, and we say also that $\lim_n \psi_n(z) = \psi(z)$ *uniformly* with respect to $z \in Z$. We note that, if $(Z) - \lim_n \psi_n = \psi$, then $\lim_n \psi_n(z) = \psi(z), \ \forall \ z \in Z$.
A consequence of 5. is the following property, which we will use in the sequel. For any set $X$, for each lattice $\Sigma \subset \mathcal{P}(X)$, for any "measure" $\mu$, for every sequence $(A_n)_n$ in $\Sigma$, such that

$$(\Sigma) - \lim_n \ \mu(A \cap A_n) = 0,$$

then

$$(\Sigma) - \lim_n \ \mu(A \cap A_n \cap B_n) = 0$$

for every $(B_n)_n$ in $\Sigma$.

For every $Z' \subset Z$, let

$$\mathcal{T}_{Z'} \equiv \{(\psi_n|_{Z'})_n : (\psi_n) \in \mathcal{T}_Z\},$$

and define $\Phi_{Z'} : \mathcal{T}_{Z'} \to R$ by setting $\Phi_{Z'}((\psi_n|_{Z'})_n) \equiv \Phi((\psi_n)_n)|_{Z'}$. This defines a $(Z')$-limit in $R$.

Let $R_1$ be an Abelian group; $R, R_2$ be convergence groups. Suppose that a "product" $\cdot : R_1 \times R_2 \to R$ is defined, such that

**p.1)** $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$

**p.2)** $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2, \ \forall \ r_i, \ s_i \in R_i \ (i = 1, 2)$.

**p.3)** For each $a \in R_1$, for all sequences $(b_n)_n$ in $R_2$, such that $\lim_n \ b_n = 0$, we have $\lim_n \ a \cdot b_n = 0$.

There are many situations in which such products arise naturally: for example, see [3].

Let now $X$ be any set, and $\Sigma \subset \mathcal{P}(X)$ be an algebra. The aim of this paper is to give a definition of "integral" of a function $f \in R_1^X$ with respect to set functions $\mu : \Sigma \to R_2$; our integral will be a $R$-valued finitely additive functional. Moreover, we will prove some Vitali-type theorems. To do this, we introduce the concepts of "convergence in $L^1$", "uniform integrability" and "convergence in measure". So, in order to speak of "limits" of integrals and of the values that $\mu$ assumes in suitable sets of $\Sigma$, we need a "convergence structure" on $R$ and $R_2$ respectively: this is the reason for which we gave the axioms 2.1 and 2.2. So, we include both topological convergence (in the case of topological groups) and $(o)$-convergence (in the case of Riesz spaces). We note that there are some Riesz spaces, for which the $(o)$-convergence is not generated by *any* topology (see also [3]).
However, we will not speak directly of "convergences" for elements of $R_1$; indeed, in order to formulate the concept of "convergence in measure", we need to "distinguish" the "very small" sets of $R_1$ from the "not too small" sets, and in general we cannot do this by means of "inequalities". So, we endow $R_1$ with a particular structure, similar to the one of "neighborhoods" of 0. This will "generate" a convergence; hence this structure is "richer" than the "convergence" above defined. Though, in Riesz spaces, the

convergence induced by this structure is in general different from the $(o)$-convergence, it turns out that it yields the "typical" concept of convergence in measure (see also [3]).

**Definition 2.3** A *theory* $\mathcal{Q}$ is a class of subsets of $R_1$, such that:

$I_1)$ $0 \in Q, \ \forall \ Q \in \mathcal{Q}$.

$I_2)$ If $Q \in \mathcal{Q}$, then $-Q \in \mathcal{Q}$.

$I_3)$ If $(Q_n)_n, (Q'_n)_n$ are in $\mathcal{Q}$, with $Q_n \downarrow \{0\} \downarrow Q'_n$, then there exists $(P_n)_n$ in $\mathcal{Q}$, with $Q_n + Q'_n \subset P_n \downarrow \{0\}$.

**Example 2.4** If $R_1$ is a $\sigma$-Dedekind complete Riesz space, we can take, as in [3],

$$\mathcal{Q} \equiv \{[-u, u] : u \in R_1, u \geq 0, u \neq 0\};$$

it is easy to see that $\mathcal{Q}$ is a theory.

If $R_1$ is a topological group, our theory $\mathcal{Q}$ will be the family of all neighborhoods of zero.

It is possible to give in $R_1$ a definition of "limit", compatible with $\mathcal{Q}$, in the following way:

**Definition 2.5** Given a sequence $(a_n)_n$ in $R_1$, and an element $a \in R_1$, we say that $\lim_n a_n = a$ if, for all $Q \in \mathcal{Q}$, there exists $n_0 \in I\!N$, such that $a_n - a \in Q, \ \forall \ n \geq n_0$.

**Definition 2.6** If $(f_n)_n$ in a sequence of elements of ${R_1}^X$, and $f \in {R_1}^X$, we say that $(X) - \lim_n f_n = f$ if, for all sequences $(x_n)_n$ in $X$, $\lim_n \ [f_n(x_n) - f(x_n)] = 0$, where the involved limit is intended in the sense of Definition 2.5.

It is easy to see that the "convergences" introduced in 2.5 and 2.6 satisfy the properties of 2.1 and 2.2 respectively, and that, in the second example of 2.4, the convergence in 2.5 coincides with the topological convergence.

In the first example of 2.4, we can associate with every $x \in R_1$ the family

$$\mathcal{Q}_x \equiv \{[x - u, x + u] : u \in R_1, u \geq 0, u \neq 0\};$$

it is easy to check that the $Q_x$, as $x$ varies in $R_1$, satisfy the properties of a neighborhood basis for some suitable topology; and so the convergence in 2.5 is a topological convergence, which implies $(o)$-convergence (see also [10]), and in general it is stronger than $(o)$-convergence.

We now give some structural assumptions, by means of which we "link" the "theory" in $R_1$ with the "convergence" in $R_3$.

$B)$ If $Z$ is any set, $(h_n)_n$ is a sequence in $R^Z$, and there exist $r \in R_2$ and $(Q_n)_n$ in $\mathcal{Q}$, with $Q_n \downarrow \{0\}$, such that

$$h_n(z) \in Q_n \cdot r \ \ \forall \ n \in I\!N, \ \forall \ z \in Z,$$

then $(Z) - \lim_n \ h_n = 0$.

$C)$ If $h \in R^Z$, and $(h_n)_n$ is a sequence in $R^Z$, such that $(Z) - \lim_n \ h_n = h$, and there exist $r \in R_2$ and $Q \in \mathcal{Q}$, such that

$$h_n(z) \in Q \cdot r \ \ \forall \ n \in I\!N, \ \forall \ z \in Z,$$

then $h(z) \in Q \cdot r, \ \ \forall \ z \in Z.$

**Example 2.7** It is easy to check that, if $R_1$ and $\mathcal{Q}$ are as in the first example of 2.4, and $R$ is a $\sigma$-Dedekind complete Riesz space, endowed with $(o)$-convergence, then $B)$ and $C)$ are fulfilled.

Indeed, we note that if $(Q_n)_n$ is a sequence of elements of $\mathcal{Q}$, $\ Q_n \equiv [-w_n, w_n]$, then $\cap_n \ Q_n = \{0\}$ if and only if $\inf_n \ w_n = 0$.

Let $R_2$, $R$ be two convergence groups, and $R_1$ be an Abelian group, endowed with a "theory" as in Definition 2.3.

**Definition 2.8** A map $\mu : \Sigma \to R_2$ is called *mean* if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A, B \in \Sigma, \ A \cap B = \emptyset$.

**Definition 2.9** Fixed a map $\mu : \Sigma \to R_2$, a map $\nu : \Sigma \to R$ is *absolutely continuous with respect to* $\mu$ if for every sequence $(A_n)_n$ in $\Sigma$ such that $(\Sigma) - \lim_n \mu(A \cap A_n) = 0$, then

$$(\Sigma) - \lim_n \nu(A \cap A_n) = 0.$$

**Definition 2.10** A sequence $(f_n)_n$ of elements of $R_1^X$ is said to be *convergent in measure* to $f$ if there exist a decreasing sequence $(Q_n)_n$ in $\mathcal{Q}$, with $\cap_n Q_n = \{0\}$, and a corresponding sequence $(E_n)_n$ in $\Sigma$, such that

$$(\Sigma) - \lim_n \ \mu(A \cap E_n) = 0,$$

and

$$\{x \in X : f_n(x) - f(x) \notin Q_n\} \subset E_n, \ \ \forall \ n \in I\!N.$$

For example, let $R_1 = R_3$ be any Banach space, $R_2 \equiv I\!R$ and $\mu : \Sigma \to R_2$ any positive mean. As $I\!R$ has Egoroff property, it is easy to check that this definition of convergence in measure is equivalent to the classical one (see also [4]). The following result holds:

**Proposition 2.11** *If $(f_n)_n$ and $(g_n)_n$ are two sequence of functions in ${R_1}^X$, convergent in measure to $f$ and $g$ respectively, then $(f_n \pm g_n)_n$ converges in measure to $f \pm g$.*

**Proof:** Straightforward.

## 3 The abstract integral.

We will construct an integral, for suitable functions $f \in {R_1}^X$ with respect to $R_2$-valued means $\mu$. Our integral will be an element of $R$.

Firstly, we define the integral for simple functions as usual; then, rather than extending it to some group of "integrable" functions, we will give a more abstract definition, by replacing the "basic" group of simple functions with a general subgroup $L$ of ${R_1}^X$, such that $f \cdot \chi_A \in L$ for every $A \in \Sigma$, whenever $f \in L$ (here, $f \cdot \chi_A$ is the function which coincides with $f$, in the points of $A$, and is 0 elsewhere). Finally, we will prove a Vitali-type theorem.

**Definition 3.1** Let $f \in R_1^X$ be a simple function, $f = \sum_{i=1}^{s} u_i \; \chi_{X_i}$, with $X_i \in \Sigma, \; \forall \; i = 1, \ldots, s$. Then we have

$$\int_X f d\mu \equiv \sum_{i=1}^{s} u_i \cdot \mu(X_i).$$

If $A \in \Sigma$, set $\int_A \; f \; d\mu \;\; \equiv \;\; \int_X \; f \; \chi_A \; d\mu$. Moreover the integral does not depend on the choice of the representation of the simple function and is additive.

The integral here introduced is absolutely continuous with respect to $\mu$ ($\int_\cdot \; f \; d\mu \ll \mu$). In fact:

**Theorem 3.2** *Let $f$ be a simple function. Suppose that $(A_n)_n$ is a sequence in $\Sigma$, such that $(\Sigma) - \lim_n \; \mu(A \cap A_n) = 0$. Then, $(\Sigma) - \lim_n \; \int_{A \cap A_n} \; f \; d\mu = 0$.*

**Proof:** Let $f \equiv \sum_{k=1}^{s} \; u_k \; \chi_{X_k}$, and pick $(A_n)_n$ such that $(\Sigma) - \lim_n \; \mu(A \cap A_n) = 0$. As

$$\int_{A \cap A_n} f \; d\mu = \sum_{k=1}^{s} \; u_k \cdot \mu(X_k \cap A \cap A_n),$$

then the assertion follows by virtue of the properties of the "product" and the convergence. □

As we stated at the beginning of this section, we will define an integral, by starting with a subgroup $L$ of ${R_1}^X$. We will assume (by *hypothesis* ) that an "integral" $I : \Sigma \times L \to R$ is defined, such that

$P_0$) If $f \in L$ is simple, then $I(A, f) = \int_A \; f \; d\mu, \; \forall \; A \in \Sigma$.

$P_1$) $I(A, \cdot)$ is additive, $\forall \; A \in \Sigma$, and

$P_2$) $I(\cdot, f)$ is a finitely additive and absolutely continuous set function, $\forall \; f \in L$.

We will now extend the subgroup $L$, by means of suitable "rules", and denote by $\mathcal{L} \subset {R_1}^X$ the "completion". We will define an extension $\tilde{I}$ of $I$, $\;\tilde{I} : \Sigma \times \mathcal{L} \to R$, still satisfying $P_1$) and $P_2$) for every $f \in \mathcal{L}$.

We will now give a condition which takes the place of "boundedness" of means.

**Definition 3.3** Let $\mu$ be a mean. We say that $\mu$ *satisfies property* $S)$ if there exists $r_0 \in R_2$ such that, for all $f \in L$ and $Q \in \mathcal{Q}$,

$$\int_A f \, d\mu \in Q \cdot r_0, \quad \forall \, A \in \Sigma, \; A \subset \{x \in X : f(x) \in Q\}.$$

It is clear that, in many classical situations, "condition $(S)$" is equivalent to "boundedness of $\mu$". From now on, we will always suppose that $\mu : \Sigma \to R_2$ satisfies $S)$.

**Definition 3.4** Let $(f_n)_n, \; f_n \in L \; \forall \; n$. We say that $(f_n)_n$ *satisfies property* $U)$ if, for each sequence $(A_n)_n$ in $\Sigma$, such that $(\Sigma) - \lim_n \; \mu(A \cap A_n) = 0$, then

$$(\Sigma) - \lim_n \int_{A \cap A_n} f_n \, d\mu = 0.$$

**Definition 3.5** We say that $(f_n)_n, \; f_n \in L$, *converges in* $L^1$ *to* $f \in L$ if

$$(\Sigma) - \lim_n \int_A (f_n - f) \, d\mu \; = 0.$$

**Theorem 3.6** *Let $f_n$ be in $L$, and assume that $(f_n)_n$ converges in measure to* $0$ *and satisfies condition $U)$. Then, $(f_n)_n$ converges in $L^1$ to* $0$.

**Proof**: Let $(Q_n)_n$ be in $\mathcal{Q}$ and $(E_n)_n$ in $\Sigma$, satisfying the definition of convergence in measure. By $U)$, we have:

$$(\Sigma) - \lim_n \int_{A \cap E_n} f_n \, d\mu = 0.$$

Let now $r_0$ be as in $S)$. We get:

$$\int_{A \cap E_n^c} f_n d\mu \in Q_n \cdot r_0$$

and so, by virtue of $B)$, we obtain:

$$\lim_n \int_{A \cap E_n^c} f_n \, d\mu = 0.$$

As

$$\int_A f_n \, d\mu = \int_{A \cap E_n} f_n \, d\mu + \int_{A \cap E_n^c} f_n \, d\mu$$

$\forall \; n \in I\!N, \; \forall \; A \in \Sigma$, then we obtain:

$$(\Sigma) - \lim_n \int_A f_n \, d\mu = 0. \quad \Box$$

We note that, in general, even in the case in which $R_1, \; R_2$ and $R$ are Riesz spaces, convergence in $L^1$ does not imply convergence in measure (see also [3]).

**Definition 3.7** A map $f \in {R_1}^X$ is said to be *integrable* w. r. to $\mu : \Sigma \to R_2$ if there exists a sequence $(f_n)_n, \; f_n \in L$, satisfying property $U)$, convergent in measure to $f$, and such that there exists in $R^\Sigma$ the limit $(\Sigma) - \lim_n \int_A \; f_n \; d\mu$. Such a sequence will be called *defining sequence for* $f$. In this case, we set, for every $A \in \Sigma$,

$$\int_A f \, d\mu \equiv \lim_n \int_A f_n \, d\mu.$$

We denote by $\mathcal{L}$ the group of all integrable functions.

Now, we prove that the integral in 3.7. is well-defined.

**Theorem 3.8** *Let $f \in {R_1}^X$ be an integrable function, and $(f_n)_n$ a defining sequence. Then the limit $(\Sigma) - \lim_n \int_A f_n \, d\mu$ does not depend on the choice of the sequence $(f_n)_n$.*

**Proof:** Let $(f_n^1)_n$, $(f_n^2)_n$ be two defining sequences for $f$, choose $A \in \Sigma$, set

$$l_i \, (A) \equiv \lim_n \int_A f_n^i \, d\mu \; (i = 1, 2),$$

and put $k_n(x) \equiv f_n^1(x) - f_n^2(x), \; \forall \, x \in X, \; \forall \, n \in I\!N$.
As $(f_n^1)_n$ and $(f_n^2)_n$ satisfy $U)$, then $(k_n)_n$ does too. Moreover, $(f_n^1)_n$ and $(f_n^2)_n$ converge in measure to $f$, and so $(k_n)_n$ converges in measure to 0. By Theorem 3.6, $k_n \to 0$ in $L^1$, that is $\lim_n \int_A k_n \, d\mu = 0$. So, we have, $\forall \, A \in \Sigma$:

$$l_1(A) - l_2(A) = \lim_n \; [l_1(A) - \int_A f_n^1 \, d\mu] +$$

$$+ \lim_n \; [\int_A f_n^2 \, d\mu - l_2(A)] + \lim_n \int_A k_n \, d\mu = 0,$$

that is the assertion. □

We observe that the definition of convergence in $L^1$ and condition $U)$ can be formulated analogously as above, even when $f_n, f \in \mathcal{L}$.

We now show that, if $f$ is integrable, then $\int_\cdot f \, d\mu \ll \mu$.

**Theorem 3.9** *Let $f \in \mathcal{L}$. If $(\Sigma) - \lim_n \; \mu(A \cap A_n) = 0$, then*

$$(\Sigma) - \lim_n \int_{A \cap A_n} f \, d\mu = 0.$$

**Proof:** Let $(f_n)_n$ be a sequence, defining $f$, and $(A_n)_n$ in $\Sigma$, such that $(\Sigma) - \lim_n \mu(A \cap A_n) = 0$. We have:

$$(\Sigma) - \lim_n \int_A (f - f_n) \, d\mu = 0,$$

and then

$$(\Sigma) - \lim_n \int_{A \cap A_n} (f - f_n) \, d\mu = 0.$$

By property $U)$, one has:

$$(\Sigma) - \lim_n \int_{A \cap A_n} f_n \, d\mu = 0.$$

Thus,

$$(\Sigma) - \lim_n \int_{A \cap A_n} f \, d\mu = (\Sigma) - \lim_n \int_{A \cap A_n} (f - f_n) \, d\mu + (\Sigma) - \lim_n \int_{A \cap A_n} f_n \, d\mu = 0. \; \square$$

We now prove that condition $S$) introduced in Definition 3.3 holds even when we replace the space $L$ with the space $\mathcal{L}$ of all "integrable" functions.

**Theorem 3.10** *Let $\mu$ and $r_0$ be as in $S$). Then, for all $f \in \mathcal{L}$ and $Q \in \mathcal{Q}$,*

$$\int_A f \; d\mu \in Q \cdot r_0, \quad \forall \; A \in \Sigma, \; A \subset \{x \in X : f(x) \in Q\}.$$

**Proof**: Let $f \in \mathcal{L}$, $Q \in \mathcal{Q}$, and pick a defining sequence $(\varphi_n)_n$ for $f$. Choose $A \in \Sigma$, such that

$$A \subset \{x \in X : f(x) \in Q\}.$$

It is easy to check that $(\varphi_n \cdot \chi_A)_n$ is a defining sequence for $f \cdot \chi_A$. Hence, by virtue of $S$) and $C$), it follows that

$$\int_A f \; d\mu = \lim_n \int_A \varphi_n \; d\mu \in Q \cdot r_0. \; \square$$

Finally, we now prove the main theorem.

**Theorem 3.11** *Let $(f_n)_n, f$ be in $\mathcal{L}$; assume that $(f_n)_n$ converges in measure to $f$ and satisfies condition $U$). Then, $(f_n)_n$ converges in $L^1$ to $f$. Conversely each sequence $(f_n)_n$ of functions in $\mathcal{L}$, convergent in $L^1$ to $f \in \mathcal{L}$, satisfies $U$).*

**Proof**: For the first part of the theorem, we observe that, without loss of generality, we can assume $f \equiv 0$. So, by virtue of theorem 3.10, the proof of the first part is completely analogous to the one of Theorem 3.6.

We now turn to the second part.

Suppose that $(f_n)_n$ in $\mathcal{L}$ converges in $L^1$ to $f \in \mathcal{L}$, and that $(\Sigma) - \lim_n \mu(A \cap A_n) = 0$. Then, for all $A \in \Sigma$ and $n \in \mathbb{N}$, we have:

$$\int_{A \cap A_n} f_n \; d\mu = \int_{A \cap A_n} (f_n - f) \; d\mu + \int_{A \cap A_n} f_n \; d\mu.$$

By convergence in $L^1$, we get:

$$(\Sigma) - \lim_n \int_{A \cap A_n} (f_n - f) \; d\mu = 0.$$

By absolute continuity of our integral, we have:

$$(\Sigma) - \lim_n \int_{A \cap A_n} f \; d\mu = 0.$$

Thus, the assertion follows. $\square$

# 4 Comparisons and applications.

## 4.1 Comparison with Bochner and Pettis integral

Let $R = R_1$ be a Banach space of infinite dimension, and $R_2 = \mathbb{R}$. Suppose that $L$ is the space of all simple functions, and, without loss of generality, assume that $\mu$ is a positive finitely additive bounded set

function.
We observe that, if $f \in R^X$ is Bochner integrable, then there exists a sequence $(f_n)_n$ of simple functions, (strongly) convergent in measure to $f$, and such that

$$\lim_n \int_A \|f_n - f\| \, d\mu = 0, \text{ uniformly w. r. to } A \in \Sigma.$$

Moreover, the sequence $(f_n)_n$ is uniformly integrable, and thus, if $(A_n)_n$ in $\Sigma$ is such that $\mu(A_n) \to 0$, then

$$\lim_n \int_{A_n} \|f_n\| \, d\mu = 0,$$

and hence

$$\lim_n \| \int_{A\cap A_n} f_n \, d\mu\| = 0 \text{ uniformly with respect to } A \in \Sigma.$$

So, every Bochner integrable function is integrable too.

However, it is easy to see that, if $R_1 = R = \mathbb{R}$, then every integrable map is Bochner integrable too.

We will give in the sequel an example of a function, which is integrable as in Definition 3.7 (where $(\Sigma)$-convergence is the uniform convergence with respect to $A \in \Sigma$), but not Bochner integrable. To do this, we need a comparison between our integral and Pettis integral.

We prove that, if $\mu$ is countably additive and $R$ is separable, then every Pettis integrable function is integrable in the sense of Definition 3.7.

Let $h \in {R_1}^X$ be Pettis integrable. Then, by virtue of Theorem 2 of Chapter II of [6], and Theorem 5.1. of [13]), $h$ is strongly measurable, and $h = g + f$, where $g$ is bounded and Bochner integrable,

$$f = \sum_{n=1}^{\infty} x_n \chi_{E_n}, \tag{1}$$

with pairwise disjoint $E_n \in \Sigma$, $\forall\, n \in \mathbb{N}$, and the series in (1) is unconditionally convergent, and

$$(P) - \int_E f \, d\mu = \sum_{n=1}^{\infty} x_n \mu(E \cap E_n), \quad \forall\, E \in \Sigma.$$

Set

$$f_n \equiv \sum_{j=1}^{n} x_j \chi_{E_j}.$$

As $\mu$ is positive, we have that

$$[\mu(A_n) \to 0] \Longrightarrow [\int_{A\cap A_n} f_k \, d\mu \to 0] \text{ uniformly w. r. to } A \in \Sigma,\ \forall\, k \in \mathbb{N}.]$$

Then, by Vitali-Hahn-Saks theorem,

$$[\mu(A_n) \to 0] \Longrightarrow [\int_{A\cap A_n} f_k \, d\mu \to 0] \text{ uniformly w. r. to } A \in \Sigma \text{ and } k].$$

and thus

$$[\mu(A_n) \to 0] \Longrightarrow [\int_{A \cap A_n} f_n \, d\mu \to 0] \text{ uniformly w. r. to } A \in \Sigma.$$

So, condition $U)$ is satisfied, uniformly with respect to $A \in \Sigma$. Moreover, as $\mu$ is positive, bounded and $\sigma$-additive, it is easy to check that, for every $\varepsilon > 0$,

$$\lim_n \mu(\{x \in X : \|f_n(x) - f(x)\| > \varepsilon\}) = 0,$$

and hence

$$\lim_n \mu(\{x \in A : \|f_n(x) - f(x)\| > \varepsilon\}) = 0$$

uniformly with respect to $A \in \Sigma$, and

$$\lim_n \mu(\{x \in A : \|f_n(x) - f_{n+p}(x)\| > \varepsilon\}) = 0$$

uniformly with respect to $p \in I\!N$ and $A \in \Sigma$. Thus, by proceeding similarly as in Theorem 3.6, it is not difficult to see that the sequence $(\int_A f_n \, d\mu)_n$ is Cauchy uniformly with respect to $A \in \Sigma$.
So, there exists in $R$ the limit $\lim_n \int_A f_n \, d\mu$, uniformly with respect to $A \in \Sigma$. Therefore, $f$ is integrable in the sense of Definition 3.7. As $g$ is Bochner integrable, then $h$ is integrable as in Definition 3.7 too. $\Box$

We now prove that every integrable function $f$ as in Definition 3.7 (where the $(\Sigma)$-convergence is the uniform convergence with respect to $A \in \Sigma$) is Pettis integrable (here, countable additivity of $\mu$ is not required).
Let $(f_n)_n$ be a sequence of functions, defining for $f$. It is easy to see that $r^* f$ is scalarly measurable, for all $r^* \in R^*$. Set

$$B^1_{R^*} \equiv \{r^* \in R^* : \|r^*\| \leq 1\}.$$

Then, for all $r^* \in R^*_1$, we have:

$$0 \leq \int_X |r^*(f_n - f)| \, d\mu \leq \sup_{r^* \in B^1_{R^*}} \int_X |r^*(f_n - f)| \, d\mu = \| \int_X (f_n - f) \, d\mu\| \to 0.$$

So, $r^* f$ is integrable, $\forall \, r^* \in B^1_{R^*}$, and hence also $\forall \, r^* \in R^*$.
So, it follows that $f$ is Pettis integrable.

We are now ready to give the quoted example.

**Example 4.1** Let $R$ be a Banach space of infinite dimension, and let $\sum_{n=1}^{\infty} y_n$ be an unconditionally convergent series, which is not absolutely convergent, namely such that $\sum_{n=1}^{\infty} \|y_n\| = \infty$. Take $X \equiv I\!N$, and define a countably additive set function $\mu : \Sigma \to I\!R_0^+$, by setting $\mu(\{n\}) = \dfrac{1}{2^n}$, $\forall \, n$. Define $f : I\!N \to R$ by putting $f(n) \equiv 2^n \, y_n$. The function $f$ is not Bochner integrable (see also [13]). For each $n \in I\!N$, let

$$f_n(j) \equiv \begin{cases} 2^j \; y_j, & \text{if } j \leq n \\ \\ 0, & \text{if } j > n \end{cases}$$

We note that $(f_n)_n$ is a defining sequence for $f$.
By proceeding analogously as above, one can check that $f$ is integrable in the sense of Definition 3.7.

## 4.2 "Weak" integral

Up to now we have constructed an integral, in which the definitions and the obtained results involve the so-called $(\Sigma)$-convergence, that is the uniform convergence with respect to $A \in \Sigma$. The introduction of this type of convergence is essentially used when we prove absolute continuity of our integral and in the second part of Theorem 3.11.
However, by replacing uniform convergence with pointwise convergence (relatively to $\Sigma$), under suitable and not too restrictive hypotheses on $\Sigma$, $R_2$ and $R$, we can construct a "weak integral", in such a way that the "pointwise" version of Theorem 3.11 holds. Actually, as we shall see, in some cases the space $\mathcal{L}$ of weakly integrable functions coincides with the one of "strongly" integrable maps, but it is endowed with a weaker "topology", which can be thought as the classical weak topology in $L^1$. So, we introduce a condition, "inspired" to the Vitali-Hahn-Saks Theorem.

**Definition 4.2** Fixed a map $\mu : \Sigma \to R_2$, we say that a map $\nu : \Sigma \to R$ satisfies property $(AC)$ if for every sequence $(A_n)_n$ in $\Sigma$ such that $\lim_n \mu(A_n) = 0$, then

$$\lim_n \nu(A_n) = 0.$$

**Definition 4.3** Fixed a map $\mu : \Sigma \to R_2$, and a sequence of maps $(\nu_n : \Sigma \to R)_n$, we say that the maps $\nu_n$ satisfy property $(UAC)$ if for every sequence $(A_n)_n$ in $\Sigma$ such that $\lim_n \mu(A_n) = 0$, then

$$\lim_n \nu_n(A_n) = 0.$$

**Definition 4.4** Let $\Sigma \subset \mathcal{P}(X)$ be an algebra, and $\mu : \Sigma \to R_2$ be a fixed mean. We say that $R$ satisfies the *Vitali-Hahn-Saks property* (VHS *property* ) *with respect to* $\Sigma$ if, for each sequence $(m_n)_n$ of means, satisfying $(AC)$, and such that there exists in $R$ the limit

$$m_0(A) \equiv \lim_n m_n(A),$$

then the means $(m_n)$ satisfy $(UAC)$ and $m_0$ has property $(AC)$.

We suppose that the conditions $B)$ and $C)$ hold, when uniform convergence is replaced by pointwise convergence, and that

$I_4$) For all sequences $(A_n)_n$, $(B_n)_n$ in $\Sigma$,

$$\lim_n \mu(A_n \cup B_n) = 0 \text{ whenever } \lim_n \mu(A_n) = 0 \text{ and } \lim_n \mu(B_n) = 0.$$

**Definition 4.5** A sequence $(f_n)_n$ of elements of $R_1^X$ is said to be *weakly convergent in measure* to $f$ if, there exist $(Q_n)_n$ in $\mathcal{Q}$, with $Q_n \downarrow 0$ and $E_n \in \Sigma$, such that

$$\lim_n \mu(A \cap E_n) = 0, \ \forall\, A \in \Sigma,$$

and

$$\{x \in X : f_n(x) - f(x) \notin Q_n\} \subset E_n.$$

**Definition 4.6** A map $f \in {R_1}^X$ is said to be *weakly integrable* w. r. to $\mu : \Sigma \to R_2$ if there exists a sequence $(f_n)_n$, $f_n \in L$, weakly convergent in measure to $f$, and such that there exists in $R^\Sigma$ the limit $\lim_n \int_A f_n \, d\mu$, $\forall\, A \in \Sigma$. In this case, we set, for every $A \in \Sigma$,

$$\int_A f \, d\mu \equiv \lim_n \int_A f_n \, d\mu.$$

**Definition 4.7** We say that the sequence $(f_n)_n$ of weakly integrable functions *weakly converges in $L^1$ to $f \in L$* if

$$\lim_n \int_A (f_n - f)\, d\mu = 0, \ \forall\, A \in \Sigma.$$

By proceeding analogously as in Theorem 3.11, one can prove the following:

**Theorem 4.8** *Let $(f_n)_n, f$ be a sequence of weakly integrable maps of $R_1^X$; assume that $(f_n)_n$ weakly converges in measure to $f$ and the maps $A \mapsto \int_A f_n \, d\mu$ satisfy condition $(UAC)$. Then, $(f_n)_n$ weakly converges in $L^1$ to $f$. Conversely, if $(f_n)_n$ is a sequence of weakly integrable functions, weakly convergent in $L^1$ to a weakly integrable function $f$, then the maps $A \mapsto \int_A f_n \, d\mu$ have property $(UAC)$.*

For example, let $R_1 = R_3$ be a Banach space, $R_2 \equiv I\!R$, $\Sigma \subset \mathcal{P}(X)$ be a $\sigma$-algebra, and $\mu : \Sigma \to I\!R$ be a positive bounded mean, hence property (VHS) holds.
We note that, in this case, the "weak integral" introduced in 4.6 and the integral in Definition 3.7 are equivalent. Indeed, let $f$ and $(f_n)_n$ be as in Definition 4.6 , and, for all $A \in \Sigma$, let $I_A(f)$ denote the quantity $\lim_n \int_A f_n \, d\mu$. As $\mu$ is positive and real-valued, we can check that $f$ is integrable in the sense of Definition 3.7, and the two involved integrals coincide. However, convergence in $L^1$ here is no longer a "strong" one: indeed, in the real-valued case it coincides with the classical weak convergence.

## 4.3 Applications: stochastic integral

Here we show that our definition 3.7 includes the so-called Ito stochastic integral.
Let $X \equiv [a, b] \subset I\!R$, $\Sigma$ be the algebra of all (disjoint) finite unions of subintervals of $[a, b]$ of the type $[\alpha, \beta[$.

Assume that $(\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{F}_t)_{t\in[a,b]}$ be a filtration of $\mathcal{F}$; $R_1 = R_2 = R = L^0(\Omega, \mathcal{F}, P)$. A map $f \in R_1^X$ is said to be *progressively measurable* if, for every $s \in [a, b]$, the map $(t, \omega) \mapsto f(t)(\omega)$ is $(\mathcal{B}[a, s] \times \mathcal{F}_s)$-measurable.
Let $L \subset R_1^X$ be the space of all functions of the type

$$f(t)(\omega) = \sum_{i=0}^{n-1} f_i(\omega)\ \chi_{[t_i,t_{i+1}[}(t),$$

where $a \equiv t_0 < t_1 < \ldots < t_n \equiv b$, $f_i \in L^0(\Omega, \mathcal{F}_{t_i}, P)\ \forall\ i = 1, \ldots, n-1$. Let $\mathcal{L} \subset R_1^X$ be the space of all progressively measurable functions $f$, such that

$$P(\int_a^b |f(\tau)(\omega)|^2\ d\tau < +\infty) = 1.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t \in [a, b], \mathcal{B}, (B_t)_{t\in[a,b]}, P)$ be a Brownian Motion, and put

$$\mu([\alpha, \beta[) \equiv B_\beta - B_\alpha,\ \forall\ a \le \alpha < \beta \le b.$$

Our "theory" of $R_1$ will be the set of all balls centered in the "origin" of $R_1$, with respect to the following "distance", equivalent to convergence in probability:

$$d(x, y) = \int_a^b \arctan |x(\omega) - y(\omega)|\ d\omega,\quad \forall\ x, y \in R_1.$$

If $g \in R_1^X$ is a simple function, we define its integral as usual, and indicate it by the symbol $\int_a^b g(t)\ dB_t$. We note that $L$ may be a proper subset of the space of the simple functions. By 6.3. of [2], for all $f \in \mathcal{L}$, there exists a sequence $(f_n)_n$ in $L$, such that $(f_n)$ is Cauchy in $L^1$, and so convergent in $L^1$. We denote by $\int_a^b f(t)\ dB_t$ the limit $\lim_n \int_a^b f_n(t)\ dB_t$. The sequence $(f_n)_n$ is defining for $f$ in the sense of Definition 3.6.

# References


[1] A. AVALLONE–A. BASILE "Integration: uniform structure", J. Math. Anal. Appl. **159** (1991), 373-381.

[2] P. BALDI "Equazioni differenziali stocastiche e applicazioni", Pitagora, Bologna (1984)

[3] A. BOCCUTO "Abstract integration in Riesz spaces", Tatra Mountains Math. Publ., **5** (1995), 107-124.

[4] A. BOCCUTO–A. R. SAMBUCINI "Comparison between different types of abstract integrals in Riesz spaces", to appear on Rend. Circ. Mat. Palermo.

[5] R. CRISTESCU–G. MARINESCU "Applications of the theory of distributions," John Wiley and Sons (1973).

[6] J. DIESTEL–J. J. UHL "Vector measures", Math. Surveys **15** (1977).

[7] H. R. FISCHER "Limesräume", Math. Ann., **137**, (1959), 269-303.

[8] I. FLEISCHER "Limit interchange for semigroup valued integrals", Publ. Math. Debrecen, **43** (1993), 239-244.

[9] A. G. KUSRAEV–S. A. MALYUGIN "Some problems in the theory of vector measures" (Russian), Nauka (1988).

[10] W. A. J. LUXEMBURG–A. C. ZAANEN "Riesz spaces", Vol I (1971), North-Holland Publishing Co.

[11] W. MARIK "Integration in Topologischen Gruppen", Ph. D. Thesis (1979).

[12] W. MARIK "Monoidwertige Integrale", Czechoslovak Math. J. **37** (1987), 351-375.

[13] K. MUSIAL "Topics in the theory of Pettis integration", School on Measure Theory and Real Analysis, Grado, 14-25 October 1991, Rend. Ist. Mat. Univ. Trieste, **23** (1991), 177-262.

[14] S. NAKANISHI, "Integration of ranked vector space valued functions", Math. Japonica **33** (1988), 105-128.

[15] E. PAP, "Comparison of $K$-convergences", Rend. Circ. Math. Palermo, Ser. II, **18** (1988), 371-380.

[16] G. PREUSS, "Semiuniform convergence spaces", Math. Japonica **41** (1995), 465-491.

[17] W. V. SMITH "Convergence in measure of integrands", Rev. Roum. Math. Pures et Appl. **26** (1981) 899-903.

[18] C. VINTI, "Martingale-Perimetro", Ann. Mat. Pura Appl., **85** (1970), 1-20.