On the De Giorgi - Letta integral with respect to means with values in Riesz spaces*

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Abstract

A monotone integral is given for scalar function, with respect to Riesz space values means, and also a necessary and sufficient condition to obtain a Radon-Nikodym density for two means.
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1 Introduction.

Integrals like Kurzweil-Stieltjes, Riemann sums and Bochner have been studied in vector lattices by Duchoň, Riečan and Vrábelová, ([11], [21], [22], Wright ([26], [27]), McGill ([19]), Šipoš ([24]), Maličký ([18]), Cristescu ([8]), Haluška ([15]), Boccuto ([3], [4]), and so on.

In this paper we extend to such spaces the monotone integral, given by Choquet in 1953 ([6]), and developped by De Giorgi-Letta ([9]), Greco ([13]), Brooks-Martellotti ([5]), and others ([10], [12], [16], etc.).

Given a mean $\mu : \mathcal{A} \to R$ and a measurable function $f : X \to \widetilde{I\!R}_0^+$, we say that f is integrable (in the monotone sense) if there exists in R the limit

$$(o) - \lim_{a \to +\infty} \int_0^a \mu(\{x \in X : f(x) > t\})\ dt.$$

For this integral we obtain some elementary properties and we give some Vitali-type theorems.
We note that in general this integral is different from the one introduced in [5] for Banach spaces.
Finally, we prove a version of Radon-Nikodym-type theorems for the introduced

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integral (see also [14]).

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2 Preliminaries.

We begin with some definitions.

Definition 2.1 A Riesz space R is called *Archimedean* if the following property holds:

2.1.1 For every choice of $a, b \in R$, if $na \leq b$ for all $n \in I\!\!N$, then $a \leq 0$.

Definition 2.2 A Riesz space R is said to be *Dedekind complete* [resp. *σ-Dedekind complete*] if every nonempty [countable] subset of R, bounded from above, has supremum in R.

The following results are well-known (see [1], [2]).

Proposition 2.3 *Every σ-Dedekind complete Riesz space is Archimedean.*

Theorem 2.4 *Given an Archimedean [Dedekind complete] Riesz space R, there exists a compact Stonian topological space Ω, unique up to homeomorphisms, such that R can be embedded as a [solid] subspace of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{I\!\!R}^\Omega : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R \ \forall \ \lambda$, and $a = \sup_\lambda \ a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \sup_\lambda \ a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{\omega \in \Omega : (\sup_\lambda \ a_\lambda)(\omega) \neq \sup_\lambda \ a_\lambda(\omega)\}$ is meager in Ω.*

Definition 2.5 A sequence $(r_n)_n$ is said to be *order-convergent* (or *(o)-convergent*) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall \ n \in I\!\!N$, and we will write $(o) - \lim_n \ r_n = r$.

As $|r_n| \leq |r| + p_1 \ \forall \ n$, every (o)-convergent sequence is bounded.
We note that, if R is a σ-Dedekind complete Riesz space, (o)-convergence can be formulated in the following equivalent ways (see also [25]):

Proposition 2.6 *A sequence $(r_n)_n$, bounded in R, (o)-converges to r if and only if*

$$r = (o) - \limsup_n \ r_n = (o) - \liminf_n \ r_n,$$

where

$$(o) - \limsup_n \ r_n = \inf_n [\sup_{m \geq n} \ r_m], \ (o) - \liminf_n \ r_n = \sup_n [\inf_{m \geq n} \ r_m].$$

Proposition 2.7 *Let R be as above, Ω as in Theorem 2.4. A bounded sequence $(r_n)_n$, $r_n \in R$, (o)-converges to r if and only if the set $\{\omega \in \Omega : r_n(\omega) \not\to r(\omega)\}$ is meager in Ω.*

We recall some fundamental properties of the order convergence (see [25]).

Proposition 2.8 *If $(r_n)_n$ (o)-converges to both r and s, then $r \equiv s$. If $(r_n)_n$ (o)-converges to r, $(s_n)_n$ (o)-converges to s , and $\alpha \in I\!\!R$, then $(r_n + s_n)_n, (r_n \vee s_n)_n, (r_n \wedge s_n)_n, (\alpha\, r_n)_n, (|r_n|)_n$ (o)-converge respectively to $r+s,\ r \vee s,\ r \wedge s,\ \alpha\, r,\ |r|$.*

Definition 2.9 A sequence $(r_n)_n$ is said to be (o)-*Cauchy* if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r_m| \le p_n,\ \forall\ n \in I\!\!N$, and $\forall\ m \ge n$.

Definition 2.10 A Riesz space R is called (o)-*complete* if every (o)-Cauchy sequence is (o)-convergent.

The following result holds (see [17], [28]):

Proposition 2.11 *Every σ-Dedekind complete Riesz space is (o)-complete.*

We note that there are some cases, in which (o)-convergence is not "generated" by a topology: for example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive non-atomic positive $\widetilde{I\!\!R}$-valued measure. We recall that, in such spaces, (o)-convergence coincides with almost everywhere convergence (see also [25]).

3 The monotone integral.

Definition 3.1 Let X be any set, R a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra. A map $\mu : \mathcal{A} \to R$ is said to be *mean* if $\mu(A) \ge 0,\ \forall\ A \in \mathcal{A}$, and $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. A mean μ is *countably additive* (or σ-*additive*) if $\mu(\cap_n\ A_n) = \inf_n\ \mu(A_n)$, whenever $(A_n)_n$ is a decreasing sequence in $\mathcal{A}$, such that $\cap_n\ A_n \in \mathcal{A}$.

Given a mapping $f : X \to \tilde{I\!\!R}_0^+$ and a mean μ as above, for all $A \in \mathcal{A}$ and $t \in I\!\!R_0^+$, set: $E_{t,A}^f$ (or simply $E_{t,A}$, when no confusion can arise) $\equiv \{x \in A : f(x) > t\}$; E_t^f $(E_t) \equiv \{x \in X : f(x) > t\}$; and, for every $t > 0$, let $u_{A,f}(t) \equiv \mu(E_{t,A}^f)$; $u_f(t) = u(t) \equiv \mu(E_t)$.

Definition 3.2 With the same notations as above, we say that a function $f : X \to \tilde{I\!\!R}_0^+$ is *measurable* if $E_t^f \in \mathcal{A},\ \forall\ t \in I\!\!R^+$.

Now, we define a Riemann [Lebesgue]-type integral, for maps, defined in an interval of the real line, and taking values in a Dedekind complete Riesz space (for similar integrals existing in the literature, see also [21] and [20]).

Definition 3.3 Let $a, b \in I\!\!R,\ a < b$, and R be as above. We say that a map $g : [a, b] \to R$ is a *step function* if there exist $n+1$ points $x_0 \equiv a < x_1 < \ldots < x_n \equiv b$, such that g is constant in each interval of the type $]x_{i-1}, x_i[$ $(i = 1, \ldots, n)$. We say that g is *simple* if there exist n elements of R, $a_1, \ldots, a_n$, and n pairwise disjoint measurable sets E_i, such that $g = \sum_{i=1}^n\ a_i\ \chi_{E_i}$. If g is a step [simple] function, we put $\int_a^b\ g(t)\ dt \equiv \sum_{i=1}^n\ (x_i - x_{i-1}) \cdot g(\xi_i)$ $[\sum_{i=1}^n\ |E_i| \cdot g(\xi_i)]$, where ξ_i is an arbitrary point of $]x_{i-1}, x_i[$ $[E_i]$.

Definition 3.4 Let $u : [a, b] \to R$ be a bounded function. We call *upper integral* [resp. *lower integral*] of u the element of R given by

$$\inf_{v \in V_u} \int_a^b v(t)\ dt \quad [\sup_{s \in S_u} \int_a^b s(t)\ dt],$$

where

$$\begin{aligned} V_u &\equiv \{v : v \text{ is a step [simple] function },\ v(t) \geq u(t),\ \forall\ t \in [a, b]\} \\ S_u &\equiv \{s : s \text{ is a step [simple] function },\ s(t) \leq u(t),\ \forall\ t \in [a, b]\}. \end{aligned}$$

We say that u is *Riemann* [*Lebesgue*] *integrable* (or (R) $[(L)]$-*integrable*), if its lower integral coincides with its upper integral, and, in this case, we call *integral of* u (and write $\int_a^b u(t)\ dt$) their common value.

It is easy to check that this integral is well-defined, and is a linear monotone functional, with values in R.
The following result holds:

Proposition 3.5 *Every bounded monotone map $u : [a, b] \to R$ is Riemann integrable.*

Proof: The proof is almost identical to the classical one.

Now, we define an integral for extended real-valued functions, with respect to R-valued means.

Definition 3.6 Let $X,\ R,\ \mu,\ f : X \to \widetilde{I\!R}_0^+,\ u = u_f$ be as above. We say that f is *integrable* if there exists in R the quantity

(3.6.1) $$\int_0^{+\infty} u(t)\ dt \equiv \sup_{a>0} \int_0^a u(t)\ dt = (o) - \lim_{a \to +\infty} \int_0^a u(t)\ dt,$$

where the integral in (3.6.1) is intended as in Definition 3.4. If f is integrable, we indicate the element in (3.6.1) by the symbol $\int_X f\ d\mu$. A measurable function $f : X \to I\!R$ is *integrable* if both f^+, f^- are integrable and, in this case, we set

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

Remark 3.7 We can extend Definition 3.6 when $\mu : \mathcal{A} \to R$ is any finitely additive bounded map. A measurable function f is *integrable* if and only if f is integrable with respect to μ^+, μ^-, where for every $A \in \mathcal{A}$

$$\begin{aligned} \mu^+(A) &\equiv \vee_{B \subset A,\ B \in \mathcal{A}}\ \mu(B), \\ \mu^-(A) &\equiv -\wedge_{B \subset A,\ B \in \mathcal{A}}\ \mu(B), \end{aligned}$$

and $\mu = \mu^+ - \mu^-$. In this case, we set

$$\int_X f\ d\mu \equiv \int_X f\ d\mu^+ - \int_X f\ d\mu^-.$$

(see also [7]).

An immediate consequence of Definition 3.6 and monotonicity of μ is the following:

Proposition 3.8 *If f is integrable, then, for each $A \in \mathcal{A}$, there exists in R the quantity*

$$\sup_{a>0} \int_0^a u_{A,f}(t)\ dt,$$

which we denote by $\int_A f\ d\mu$.

Proposition 3.9 *With the same notations as above, if f is integrable, then*

$$\int_A f\ d\mu = \int_X f \cdot \chi_A\ d\mu,$$

$\forall\ A \in \mathcal{A}$.

Proof: For each fixed $t > 0$, and $x \in X$, we have $[f \cdot \chi_A(x) > t]$ if and only if $[x \in A]$ and $[f(x) > t]$. So, $u_{X,f\cdot\chi_A} \equiv u_{A,f}$. Thus, the assertion follows. □

It is easy to check that this integral is a linear R-valued functional, and that, for every positive integrable map f, $\int_\cdot f\ d\mu$ is a mean.

We now list a number of technical results.

Proposition 3.10 *If f is integrable, then $(o)-\lim_{t\to+\infty}\ \mu(E_t) = 0$, and hence $\mu(E_\infty) = 0$, where $E_\infty \equiv \{x \in X : f(x) = +\infty\}$.*

Proof: For every $t > 0$, we have:

$$0 \le \mu(E_\infty) \le \mu(E_t) = \frac{\int_{E_t}\ t\ d\mu}{t} \le \frac{\int_{E_t}\ f\ d\mu}{t} \le \frac{\int_X\ f d\mu}{t}.$$

Taking the infimum, we obtain:

$$0 \le \mu(E_t) \le \inf_{t>0} \frac{\int_X\ f d\mu}{t} = 0. \square$$

Proposition 3.11 *Let $f : X \to \tilde{I\!R}_0^+$ be measurable. Then, f is integrable if and only if*

$$\sup_n \int_X\ (f \wedge n)\ d\mu \in R,$$

and in this case

$$\sup_n \int_X\ (f \wedge n)\ d\mu = \int_X\ f\ d\mu.$$

Proof: Fix $n \in I\!N$, and pick $t < n$: then, $f(x) \wedge n > t$ if and only if $f(x) > t$, and so

$$\int_0^n\ u_f(t)\ dt = \int_0^n\ u_{f\wedge n}(t)\ dt = \int_0^{+\infty}\ u_{f\wedge n}(t)\ dt = \int_X\ (f \wedge n)\ d\mu.$$

So, the first part of the assertion follows immediately. Moreover, taking the suprema, we get

$$\sup_n \int_X\ (f \wedge n)\ d\mu = (o) - \lim_{n\to+\infty} \int_0^n\ u_f(t)\ dt = \int_X\ f\ d\mu. \square$$

Proposition 3.12 *Let $f : X \to I\!R_0^+$ be measurable and bounded, and set $S_f[V_f] \equiv \{g : X \to I\!R : g \le f,\ g$ is simple $\}$ $[\{h : X \to I\!R : h \ge f,\ h$ is simple $\}]$.*
Then, $\int_X f\ d\mu = \sup_{g \in S_f} \int_X g\ d\mu = \inf_{h \in V_f} \int_X h\ d\mu$, and f is integrable.

Proof: Without restriction, it will be enough to prove the part involving S_f.
Let $L = \sup_{x \in X} f(x)$ and, for every fixed $n \in I\!N$, let $s_n(0) \equiv u(0)$, and

$$s_n(t) \equiv u(\frac{L\ i}{2^n}),$$

whenever $t \in]\frac{L(i-1)}{2^n}, \frac{L\ i}{2^n}]$ $(i = 1, \ldots, 2^n)$. We have:

$$\int_0^L s_n(t)\ dt = \sum_{i=1}^{2^n} \frac{L}{2^n}\ u(\frac{L}{2^n}\ i).$$

Put

$$U_i^{(n)} \equiv \{x \in X : f(x) > \frac{L\ i}{2^n}\};$$

$$g_n \equiv \sum_{i=1}^{2^n} \frac{L}{2^n}\ \chi_{U_i^{(n)}}, \forall\ n \in I\!N, i = 1, 2, \ldots, 2^n.$$

Then (see also [9]):

$$\int_X g_n\ d\mu = \sum_{i=1}^{2^n} \frac{L}{2^n}\ \mu(U_i^{(n)}) = \sum_{i=1}^{2^n} \frac{L}{2^n}\ u(\frac{L}{2^n}\ i).$$

Taking the supremum, we get

$$\int_X f\ d\mu = \int_0^L u(t)\ dt = \sup_n \int_X g_n\ d\mu = (o) - \lim_n \int_X g_n\ d\mu.$$

If $g \in S_f$, then

$$\int_X g\ d\mu \le \int_X f\ d\mu,$$

and so

$$\int_X f\ d\mu = \sup_{n \in I\!N} \int_X g_n\ d\mu \le \sup_{g \in S_f} \int_X g\ d\mu \le \int_X f\ d\mu,$$

that is the assertion. □

Proposition 3.13 *If $f : X \to \widetilde{I\!R}_0^+$ is integrable, then*

$$\int_X f\ d\mu = \sup_{g \in S_f} \int_X g\ d\mu\ .$$

Conversely, if $f \ge 0$ is such that the quantity $\sup_{g \in S_f} \int_X g\ d\mu$ exists in R, then f is integrable, and

$$\int_X f\ d\mu = \sup_{g \in S_f} \int_X g\ d\mu.$$

Proof. The assertion follows by Propositions 3.11 and 3.12.

The following result is easy too:

Proposition 3.14 *Let $f : X \to \widetilde{I\!R}_0^+$ be an integrable map, $g : X \to \widetilde{I\!R}_0^+$ measurable, such that*

$$0 \leq g(x) \leq f(x), \ \forall \ x \in X.$$

Then g is integrable, and $\int_X g \ d\mu \leq \int_X f \ d\mu$.

Now, we note that, if $\mu : X \to R$ is a mean, and $\mathcal{C}_\infty(\Omega)$ is as in Theorem 2.4, then there exists a nowhere dense set $\Omega' \subset \Omega$, such that $\mu(A)(\omega)$ is real, $\forall \ \omega \notin \Omega'$, $\forall \ A \in \mathcal{A}$.

Proposition 3.15 *Let $R \subset \mathcal{C}_\infty(\Omega)$ a Dedekind complete Riesz space, where Ω' is as above, and set $\mu_\omega(A) \equiv \mu(A)(\omega)$, $\forall \ \omega \notin \Omega'$. Assume that $f : X \to I\!R$ is an integrable map. Then, there exists a meager set set $N \subset \Omega$, such that f is integrable with respect to μ_ω, and*

$$\int_A f \ d\mu_\omega = \left(\int_A f \ d\mu \right)(\omega), \ \forall \ \omega \in N^c, \forall \ A \in \mathcal{A}.$$

Proof. Without loss of generality, we can assume that f is nonnegative. Firstly, suppose that f is bounded. There exists a sequence of simple functions $(s_n)_n$ such that $s_n \uparrow f$ and $\int s_n d\mu \uparrow \int f d\mu$. So, we have, for every $n \in I\!N$, up to the complement of a meager set, depending only on X:

$$\begin{aligned} 0 \ &\leq \ \left| \int_A f d\mu_\omega - \left(\int_A f d\mu \right)(\omega) \right| \leq \left| \int_A f d\mu_\omega - \int_A s_n d\mu_\omega \right| + \\ &+ \ \left| \int_A s_n d\mu_\omega - \left(\int_A f d\mu \right)(\omega) \right| = \left| \int_A f d\mu_\omega - \int_A s_n d\mu_\omega \right| + \\ &+ \ \left| \left(\int_A s_n d\mu \right)(\omega) - \left(\int_A f d\mu \right)(\omega) \right| \leq \int_X f - s_n d\mu_\omega + \left(\int_X f - s_n d\mu \right)(\omega) \end{aligned}$$

Then:

$$\begin{aligned} 0 \ &\leq \ \left| \int_A f d\mu_\omega - \left(\int_A f d\mu \right)(\omega) \right| \leq \limsup_n \int_X f - s_n d\mu_\omega + \limsup_n \left(\int_X f - s_n d\mu \right)(\omega) = \\ &= \ \inf_n \int_X f - s_n d\mu_\omega + \inf_n \left(\int_X f - s_n d\mu \right)(\omega) = 0. \end{aligned}$$

Assume now that f is integrable. By the previous step, there exists a meager set N^* such that, $\forall \ n \in I\!N$, $\forall \ \omega \notin N^*$, $\forall \ A \in \mathcal{A}$, it holds:

$$\int_A (f \wedge n) d\mu_\omega = \left(\int_A f \wedge n \ d\mu \right)(\omega).$$

The proof is now analogous to the first part: it will be enough to replace s_n with $f \wedge n$. $\square$

Now, we prove the following:

Theorem 3.16 *Let $f : X \to \tilde{I\!R}_0^+$ be an integrable map. Then, there exists a meager set N such that, for every $A \in \mathcal{A}$, and for every $\omega \notin N$, $\left(\int_A f \, d\mu\right)(\omega) \in (\mu(A) \ \overline{co}\{f(x) : x \in A\})(\omega)$.*

Proof. By Proposition 3.15 and classical results, we have, up to the complement of a meager set:

$$(\int_A f \, d\mu)(\omega) = \int_A f \, d\mu_\omega \in \mu_\omega(A) \ \overline{co}\{f(x), x \in A\} =$$

$$\overline{co}\{f(x) \ \mu_\omega(A), x \in A\} = (\mu(A) \ \overline{co}\{f(x), x \in A\})(\omega).\Box$$

For the definition of absolute continuity and related remarks, see ([4]).

Proposition 3.17 *If $f : X \to \tilde{I\!R}_0^+$ is integrable, then the integral $\int_\cdot f \, d\mu$ is absolutely continuous, that is, $(o) - \lim_n \int_{A_n} f \, d\mu = 0$ whenever $(A_n)_n$ is a sequence in $\mathcal{A}$, such that $(o) - \lim_n \ \mu(A_n) = 0$.*

Proof: The assertion is trivial when f is bounded. So, we prove absolute continuity in the general case. Fix $n, k \in I\!N$, and pick $(A_n)_n$, with $(o) - \lim_n \mu(A_n) = 0$. We have:

$$\begin{aligned} 0 \ &\leq \ \int_{A_n} f \, d\mu = \int_{A_n} (f \wedge k) \, d\mu + \int_{A_n} f - (f \wedge k) \, d\mu \leq \\ &\leq \ \int_{A_n} (f \wedge k) \, d\mu + \int_X f - (f \wedge k) \, d\mu. \end{aligned}$$

As $(o) - \lim_k \int_X f - (f \wedge k) \, d\mu = 0$, and $(o) - \lim_n \int_{A_n} (f \wedge k) \, d\mu = 0$ for each $k \in I\!N$, then there exist a sequence $(r_k)_k$ in R, $r_k \downarrow 0$, and a double sequence $(r'_{n,k})_{n,k}$ in R, $r'_{n,k} \downarrow 0$ $(n \to +\infty, k = 1, 2, \ldots)$, such that

$$0 \leq \int_{A_n} f \, d\mu \leq r'_{n,k} + r_k, \quad \forall \ n, k \in I\!N.$$

It follows that

$$0 \leq (o) - \limsup_{n \to +\infty} \int_{A_n} f \, d\mu \leq ((o) - \limsup_{n \to +\infty} \ r'_{n,k}) + r_k = r_k, \quad \forall \ k \in I\!N.$$

By arbitrariness of k, we get:

$$(o) - \limsup_{n \to +\infty} \int_{A_n} f \, d\mu = 0,$$

and hence

$$(o) - \lim_{n \to +\infty} \int_{A_n} f \, d\mu = 0. \ \Box$$

Now, we will prove a Vitali-type theorem for our integral.

Definition 3.18 Let $(f_n : X \to \widetilde{I\!R})_n$ be a sequence of integrable functions. We say that $(f_n)_n$ is *uniformly integrable* if

$$\sup_n \int_X |f_n| \, d\mu \in R, \tag{1}$$

and

$$(o) - \lim_n \sup_{k \geq n} \left(\int_{A_n} |f_k| \, d\mu \right) = 0, \tag{2}$$

whenever $(o) - \lim_k \ \mu(A_k) = 0$.

Definition 3.19 Under the same hypotheses and notations as above, we say that $(f_n)_n$ *converges in* L^1 *to* f if

$$(o) - \lim_n \int_X |f_n - f| \, d\mu = 0.$$

Remark 3.20 It is easy to check that $(f_n)_n$ converges in L^1 to f if and only if

$$\int_A f \, d\mu = (o) - \lim_{n \to +\infty} \int_A f_n \, d\mu$$

uniformly with respect to $A \in \mathcal{A}$.

Theorem 3.21 [Vitali 's theorem]. *Under the same notations as above, let* $(f_n)_n$ *be a uniformly integrable sequence of functions, convergent in measure to* f*. Then,* f *is integrable, and* $(f_n)_n$ *converges in* L^1 *to* f*.*
Conversely, every sequence (f_n) *of integrable functions, convergent in* L^1 *to an integrable map* f*, is convergent in measure to* f *and uniformly integrable.*

Proof: To obtain the integrability of $|f|$, it is enough to prove that

$$\sup \ S_{|f|} \equiv \sup \ \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq |f| \text{ and } \varphi \text{ is simple} \right\} \in R, \tag{3}$$

by virtue of Proposition 3.13. Let $\varphi \in S_{|f|}, \varphi = \sum_{j=1}^k \ c_j \ \chi_{B_j}$. Fix $j = 1, 2, \ldots, k$, and, for every $n \in I\!N$, set $A_n \equiv E_1^{|f - f_n|}$. If $x \in {A_n}^c \cap B_j$, we have:

$$\varphi(x) = c_j \leq |f_n(x)| + 1,$$

and hence

$$\int_{B_j \cap A_n^c} \varphi(x) \, d\mu \leq \int_{B_j} |f_n(x)| \, d\mu + \mu(B_j).$$

As to $A_n \cap B_j$, we have

$$\int_{B_j \cap A_n} \varphi(x) \, d\mu \leq c_j \ \mu(A_n).$$

Thus,

$$\begin{aligned}\int_{B_j} \varphi(x)\, d\mu &\le \int_{B_j} |f_n(x)| d\mu + \mu(B_j) + c_j\, \mu(A_n),\\ \int_X \varphi(x) d\mu &\le \int_X |f_n(x)| d\mu + \mu(X) + \mu(A_n) \sum_{j=1}^k c_j.\end{aligned}$$

By convergence in measure, $(o)-\lim_{n\to+\infty} \mu(A_n)\sum_{j=1}^k c_j = 0$, and by arbitrariness of n,

$$\int_X \varphi d\mu \le \sup_n \int_X |f_n| d\mu + \mu(X) \in R.$$

Since the right hand side does not depend on φ, (3) follows.
So, $|f|$ is integrable. By Proposition 3.14, f^+ and f^- are integrable, and so is f.

Fix now $\varepsilon > 0$ and $n \in I\!N$. As f_n is integrable by hypothesis, then $f - f_n$ is too. We have:

$$\begin{aligned}\int_X |f_n - f|\, d\mu &\le \int_{\{x\in X: |f_n - f| \le \varepsilon\}} |f_n - f|\, d\mu + \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f_n - f|\, d\mu \le\\ &\le \int_X \varepsilon\, d\mu + \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f_n|\, d\mu + \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f|\, d\mu \le\\ &\le \varepsilon \cdot \mu(X) + \sup_{k\ge n} \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f_k|\, d\mu + \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f|\, d\mu.\end{aligned}$$

As $(o)-\lim_n\ \mu(\{x \in X : |f - f_n| > \varepsilon\}) = 0$, then, by virtue of uniform integrability of $(f_k)_k$, integrability of f and absolute continuity of the integral, we get

$$(o) - \lim_{n\to+\infty} [\sup_{k\ge n} \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f_k|\, d\mu + \int_{\{x\in X: |f_n - f| > \varepsilon\}} |f|\, d\mu] = 0.$$

So, there exists a sequence $(r_n)_n$ in R, $r_n \downarrow 0$, such that

$$0 \le \int_X |f_n - f|\, d\mu \le \varepsilon \cdot \mu(X) + r_n,\ \forall\, n \in I\!N.$$

Thus, we obtain:

$$\begin{aligned}0 &\le (o) - \limsup_{n\to+\infty} \int_X |f_n - f|\, d\mu \le \varepsilon \cdot \mu(X) + (o) - \limsup_{n\to+\infty}\ r_n =\\ &= \varepsilon \cdot \mu(X) + \inf_{n\in I\!N}\ r_n = \varepsilon \cdot \mu(X).\end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we get

$$(o) - \lim_{n\to+\infty} \int_X |f_n - f|\, d\mu = 0.$$

Conversely, suppose that $(f_n)_n$ converges in L^1 to f.

Fix $\varepsilon > 0$, and set

$$E_\varepsilon^{|f-f_n|} \equiv \{x \in X : |f_n(x) - f(x)| > \varepsilon\}, \ \forall \ n \in I\!N.$$

Then,

$$\frac{\int_X \ |f_n - f| \ d\mu}{\varepsilon} \geq \frac{\int_{E_\varepsilon^{|f-f_n|}} \ |f_n - f| \ d\mu}{\varepsilon} \geq \mu(E_\varepsilon^{|f-f_n|}) \geq 0,$$

and hence $(o) - \lim_n \ \mu(E_\varepsilon^{|f-f_n|}) = 0$.

Now, we prove uniform integrability. By convergence in L^1, it follows immediately that $\sup_k \ \int_X \ |f_k| \ d\mu \in R$.

Let $(A_n)_n$ be a sequence in $\mathcal{A}$, such that $(o) - \lim_n \ \mu(A_n) = 0$. Fix $n \in I\!N$. For every $k \geq n$, we have:

$$\int_{A_n} \ |f_k| \ d\mu \leq \int_{A_n} \ |f_k - f| \ d\mu + \int_{A_n} \ |f| \ d\mu \leq$$

$$\leq \int_X \ |f_k - f| \ d\mu + \int_{A_n} \ |f| \ d\mu.$$

By convergence in L^1, there exists a sequence $(r_k)_k$ in R, $r_k \downarrow 0$, such that

$$\int_X \ |f_k - f| \ d\mu \leq r_k \leq r_n.$$

Thus,

$$\sup_{k \geq n} \int_{A_n} \ |f_k| \ d\mu \leq r_n + \int_{A_n} \ |f| \ d\mu.$$

So,

$$0 \leq (o) - \limsup_{n \to +\infty} \ \sup_{k \geq n} \int_{A_n} \ |f_k| \ d\mu \leq \ \inf_n \ r_n + (o) - \limsup_{n \to +\infty} \int_{A_n} \ |f| \ d\mu = 0,$$

and hence

$$(o) - \lim_{n \to +\infty} \ \sup_{k \geq n} \int_{A_n} \ |f_k| \ d\mu = 0. \Box$$

A consequence of Vitali' s theorem is the following:

Theorem 3.22 [Lebesgue dominated convergence theorem] *Let $(f_n)_n$, f_n be a sequence of measurable functions, and suppose that there exists an integrable map h, such that $|f_n(x)| \leq |h(x)|$ for all $n \in I\!N$ and almost everywhere with respect to x. Furthermore, assume that $(f_n)_n$ converges in measure to f. Then, for every $n \in I\!N$, f_n is integrable and $(f_n)_n$ converges in L^1 to f.*

Proof: Without loss of generality, we suppose that

$$|f_n(x)| \leq |h(x)|, \ \forall\, n \in I\!N, \ \forall\, x \in X.$$

By integrability of $|h|$ and Proposition 3.14, f_n is integrable for every $n \in I\!N$; moreover, by virtue of absolute continuity of the integral of h, the hypotheses of Theorem 3.21 hold. So, the assertion follows. □

As a consequence of Theorem 3.22, we prove the following theorem, that is a sufficient condition for the convergence in L^1, inspired by a well-known result of Scheffé 's ([23]):

Theorem 3.23 *With the same notations as above, let $(f_n)_n : X \to \widetilde{I\!R}_0^+$ be a sequence of integrable functions, convergent in measure to a nonnegative integrable mapping f. Assume that $\int_X f_n \, d\mu$ (o)-converges to $\int_X f \, d\mu$. Then, $(f_n)_n$ converges in L^1 to f.*

Proof: Let $h_n(x) = f_n(x) - f(x), \ \forall\, x \in X$. Thus,

$$0 \leq [h_n(x)]^- \leq f(x), \quad \forall\, x.$$

Let $H_n(x) = [h_n(x)]^-, \ \forall\, x$. Then, $f, \ H_n$ are integrable for every n, and $(H_n)_n$ converges in measure to 0. By Theorem 3.22, we have:

$$0 = (o) - \lim_n \int_X [h_n(x)]^- \, d\mu$$

and so

$$(o) - \lim_n \int_X [h_n(x)]^+ \, d\mu = (o) - \lim_n \int_X h_n \, d\mu = 0,$$

by hypothesis.
Finally, we get:

$$\begin{aligned} (o) - \lim_n \int_X |h_n| \, d\mu &= (o) - \lim_n \int_X [h_n(x)]^+ \, d\mu + \\ &+ (o) - \lim_n \int_G [h_n(x)]^- \, d\mu = 0. \ \square \end{aligned}$$

We now state a version of the monotone convergence theorem.

Theorem 3.24 *With the same notations as above, let $(f_n)_n$ be an increasing sequence of non negative integrable maps, convergent in measure to an integrable function f. Then,*

$$\int_X f \, d\mu = (o) - \lim_n \int_X f_n \, d\mu,$$

and therefore $f_n \to f$ in L^1.

Proof: It is an immediate consequence of Vitali 's Theorem.

4 Countable additive case.

If μ is countably additive, convergence almost everywhere implies convergence in measure; this can be proved along classical lines, hence we simply state the results. So both Levi's theorem and Fatou's lemma hold.

Proposition 4.1 *Let R be a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra, and assume that $\mu : \mathcal{A} \to R$ is a σ-additive mean. Set*

$$A_n^\varepsilon \equiv \{x \in X : |f_n(x) - f(x)| > \varepsilon\}, \ \forall\, \varepsilon > 0.$$

Then, f_n converges almost everywhere to f if and only if $\mu(\limsup_n \ A_n^\varepsilon) = 0, \ \forall\, \varepsilon >$ 0.

It is easy to prove the following:

Proposition 4.2 *Let R, $\mathcal{A}$ and μ be as above, and assume that μ is σ-additive. Then, for each sequence (A_n) in $\mathcal{A}$, one has:*

$$\mu(\liminf_n \ A_n) \leq \liminf_n \ \mu(A_n) \leq \limsup_n \ \mu(A_n) \leq \mu(\limsup_n \ A_n).$$

A straightforward consequence of Proposition 4.2 is the following:

Theorem 4.3 *Let f_n, f and μ be as above. If (f_n) converges to f almost everywhere, then (f_n) converges to f in measure.*

From Theorems 3.24 and 4.3, and by proceeding as in the classical case, it follows:

Theorem 4.4 *With the same notations and hypotheses as above, let $(f_n)_n$ be an increasing sequence of nonnegative measurable maps. Then $f(x) \equiv \lim_n \ f_n(x)$ is integrable if and only if $\lim_n \int_X \ f_n \ d\mu \in R$, and in this case*

$$\int_X \ f \ d\mu = (o) - \lim_n \ \int_X \ f_n \ d\mu.$$

A consequence of Beppo Levi 's Theorem is the following version of Fatou's Lemma:

Theorem 4.5 *Let X, R, μ be as above, $(f_n)_n$ a sequence of nonnegative integrable maps, $f(x) \equiv \liminf_n \ f_n(x), \ \forall\, x \in X$. If $\liminf_n \int_X \ f_n \ d\mu \in R$, then f is integrable, and $\liminf_n \int_X \ f_n \ d\mu \geq \int_X \ f \ d\mu$.*

5 Radon-Nikodym Theorem.

In this section, we give a Greco-type condition for the existence of a Radon-Nikodym derivative for the monotone integral, introduced in the previous section (see [14]) . We show that the Radon-Nikodym problem, in general, has no solutions. Indeed, there exist two $\mathbb{R}^2$-valued σ-additive means μ and ν, with $\nu \ll \mu$, such that there is no function $f : X \equiv \{0,1\} \to \mathbb{R}$ such that $\nu = \int_X \; f \; d\mu$.
Let $X \equiv \{0,1\}$, $\mathcal{A} \equiv \mathcal{P}(X)$, $R \equiv \mathbb{R}^2$ (endowed with componentwise ordering), $\mu, \nu : \mathcal{P}(X) \to \mathbb{R}^2$ defined by setting

$$\mu(\{0\}) = (1,0), \;\; \mu(\{1\}) = (0,1), \;\; \nu(\{0\}) = (0,1), \;\; \nu(\{1\}) = (1,0).$$

It is easy to check that μ and ν are σ-additive, ν is absolutely continuous w. r. to μ and μ is absolutely continuous w. r. to ν. However, there is no function $f : X \to \mathbb{R}$, such that $\nu(A) = \int_A \; f \; d\mu, \; \forall \; A \in \mathcal{P}(X)$: otherwise, we have:

$$(1,0) = \nu(\{1\}) = \int_{\{1\}} \; f \; d\mu = f(1) \; \mu(\{1\}) = (0, f(1)),$$

contradiction.
Furthermore, it is easy to see that, for every $r > 0$, there exists no Hahn decomposition for the map $\nu - r \; \mu$.

Now we introduce two preliminary lemmas.

Proposition 5.1 *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. If there exists an $\mathcal{A}$-measurable function $f : X \to \widetilde{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$:*

$$\nu(E) = \int_E f d\mu$$

then, for every $r > 0$, the set $A_r = \{x \in X : f(x) > r\}$ satisfies:

5.1.1) *$\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$;*

5.1.2) *$\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap calA$;*

5.1.3) *$(o) - \lim_{r \to +\infty} \nu(A_r) = 0$.*

Proof: $A_r \in \mathcal{A}$ for every $r > 0$ since f is measurable; moreover, for every $r > 0$ and for every $E \in A_r \cap \mathcal{A}, F \in A_r^c \cap \mathcal{A}$, we have:

$$\nu(E) = \int_E f d\mu \geq \int_E r d\mu = r\mu(E)$$
$$\nu(F) = \int_F f d\mu \leq \int_F r d\mu = r\mu(F).$$

This proves (5.1.1) and (5.1.2).
(5.1.3) is a consequence of (5.1.1): indeed, (5.1.1) yields

$$\mu(A_r) \leq \frac{\nu(A_r)}{r} \leq \frac{\nu(X)}{r}, \; \forall \; r > 0.$$

So, $(o) - \lim_{r \to +\infty} \; \mu(A_r) = 0$, and hence $(o) - \lim_{r \to +\infty} \; \nu(A_r) = 0$. □

Proposition 5.2 *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Let $D \equiv \{\frac{i}{2^n}, i, n \in I\!N\}$. If there exists a decreasing family $(A_r)_{r\in D}$, such that $A_0 = X$ and satisfying* (5.1.1) and (5.1.2)*, then the function $f : X \to [0, +\infty]$, defined by $f(x) \equiv \sup\{r \in D : x \in A_r\}$, is integrable and*

$$\nu(E) = \int_E f d\mu, \ \forall \ E \in \mathcal{A}.$$

Proof: f is $\mathcal{A}$-measurable, since, $\forall \ t > 0$, $\{x \in X : f(x) > t\} = \cup_{r\in D, r>t} A_r$. Let $f_n \equiv \frac{1}{2^n}\sum_{k=1}^{n2^n} \chi A_{\frac{k}{2^n}}$, for every $n \in I\!N$. Then

$$f \wedge n - f \wedge \frac{1}{2^n} \leq f_n \leq f, \ \forall \ n.$$

By construction, for every $E \in \mathcal{A}$,

$$\begin{aligned}\int_E f_n d\mu &= \frac{1}{2^n}\sum_{k=1}^{n2^n} \mu(A_{\frac{k}{2^n}}) = \sum_{k=1}^{n2^n-1} \frac{k}{2^n}\left[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\right] + n\mu(A_n \cap E) \leq \\ &\leq \sum_{k=1}^{n2^n-1}\left[\nu(A_{\frac{k}{2^n}} \cap E) - \nu(A_{\frac{k+1}{2^n}} \cap E)\right] + n\nu(A_n \cap E) \leq \nu(E).\end{aligned}$$

So,

$$\sup_n \int_X f_n d\mu \leq \nu(X) \in R$$

and thus

$$\sup_n \int_X (f \wedge n) \ d\mu \leq \sup_n \int_X (f_n + 1) \ d\mu \leq \nu(X) + \mu(X).$$

So, by Proposition 3.11, f is integrable, and hence, by Proposition 3.8, $f \cdot \chi_E$ is integrable, $\forall \ E \in \mathcal{A}$. Thus

$$(o) - \lim_n \ [\int_E (f \wedge n) \ d\mu - \int_E (f \wedge \frac{1}{2^n}) \ d\mu] = (o) - \lim_n \ \int_E (f \wedge n) \ d\mu = \int_E f \ d\mu,$$

and therefore

$$(o) - \lim_n \ \int_E f_n \ d\mu = \int_E f \ d\mu$$

and

$$\int_E f d\mu \leq \nu(E), \ \forall \ E \in \mathcal{A}.$$

On the other hand,

$$\begin{aligned}\int_E f_n d\mu &= \sum_{k=1}^{n2^n-1} \frac{k+1}{2^n}\left[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\right] + n\mu(A_n \cap E) + \\ &- \frac{1}{2^n}\sum_{k=1}^{n2^n-1}\left[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\right] \geq \\ &\geq \nu(A_{\frac{1}{2^n}} \cap E) - \nu(A_n \cap E) - \frac{1}{2^n}\left(\mu(A_{\frac{k}{2^n}}) - \mu(A_n \cap E)\right).\end{aligned}$$

Taking the (o)-limits as $n \to \infty$, we obtain

$$\int_E f d\mu = \nu(E). \quad \square$$

A consequence of Proposition 5.1 and 5.2 is the following Radon-Nikodym Theorem.

Theorem 5.3 *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:*

(5.3.a) *there exists an $\mathcal{A}$-measurable function $f : X \to \widetilde{I\!R}_0^+$ such that, for every $E \in \mathcal{A}$:*

$$\nu(E) = \int_E f d\mu;$$

(5.3.b) *there exists a family $(A_r)_{r>0}$ of measurable sets such that for every $r > 0$:*

(5.3.b.1) *$\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$;*

(5.3.b.2) *$\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$.*

The following is a different formulation of 5.3.

Theorem 5.4 *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:*

(5.4.a) *there exists a $\mathcal{A}$-measurable function $f : X \to \widetilde{I\!R}_0^+$ such that, for every $E \in \mathcal{A}$:*

$$\nu(E) = \int_E f d\mu;$$

(5.4.b) *for every $r > 0$ the measure $\nu - r\mu$ admits a Hahn decomposition, namely there exist two disjoint measurable sets (B_r, C_r) such that, $\forall\ E \in \mathcal{A}$:*

$$\begin{aligned} (\nu - r\mu)^+(E) &= (\nu - r\mu)(E \cap B_r) \\ (\nu - r\mu)^-(E) &= (\nu - r\mu)(E \cap C_r) \end{aligned}$$

Proof: (5.4.a) $\Longrightarrow$ (5.4.b)
By Theorem 5.3, there exists a family $(A_r)_{r>0}$ of measurable sets such that, for every $r > 0$:

(5.3.b.1) $\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$;

(5.3.b.2) $\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$

Set $B_r \equiv A_r, C_r \equiv A_r^c$. For every $E \in A_r \cap \mathcal{A}$, we have:

$$\begin{aligned} (\nu - r\mu)^+(E) &= (\nu - r\mu)^+(E \cap A_r) + (\nu - r\mu)^+(E \cap A_r^c) = \\ &= (\nu - r\mu)^+(E \cap A_r) = (\nu - r\mu)(E \cap A_r) \end{aligned}$$

from (5.3.b.1), since $(\nu - r\mu)(F) \leq 0,\ \forall\ F \in E \cap A_r^c \cap \mathcal{A}$.
So we obtain, for every $E \in \mathcal{A}$,

$$(\nu - r\mu)^+(E) = (\nu - r\mu)(E \cap B_r).$$

Analogously, for each $E \in \mathcal{A}$,

$$(\nu - r\mu)^-(E) = (\nu - r\mu)(E \cap C_r).$$

(5.4.b) $\Longrightarrow$ (5.4.a)
It is easy to check that, if (5.4.b) holds, then (5.3.b.1.) and (5.3.b.2.) are satisfied. The assertion follows by Proposition 5.2. □

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