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**Convergence and Fubini Theorems for Metric Semigroup-Valued Functions Defined on Unbounded Rectangles**


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**Abstract**

We introduce here a version of $KH$-integral for two-variable functions with values in metric semigroups. We obtain for it convergence results and a version of the Fubini Theorem.



**AMS (MOS) Subject Classification:** 28B15

**Key words:** Metric semigroup, Kurzweil-Henstock integral, convergence theorems, Hake theorem, Fubini theorem.

# 1 Introduction

In [1, 5, 14] some Kurzweil-Henstock, Stieltjes-type integrals were investigated for functions, defined in (possibly unbounded) subintervals of the extended real line, and with values in metric semigroups. Particular cases of such structures were also studied, for example, in [15, 16, 17] (see also the bibliography of [5]).

In this paper we introduce the two dimensional Kurzweil-Henstock integral for metric semigroup-valued functions, defined in (not necessarily bounded) subrectangles of the extended Cartesian plane. We prove for it convergence results both with respect to sequences of functions (convergence theorems related with equiintegrability), and with respect to increasing families of sets (the Hake theorem). Moreover, following a line of research on double integration, see for example [2, 12, 13] in the context of Riesz spaces, we give also a version of the Fubini theorem which generalizes a similar result proved in [14] for mappings defined in a *compact* subrectangle of $\mathbb{R}^2$.

For other related results and studies existing in the literature about these types of integrals, see also [3, 4, 8, 9, 10, 11].

# 2 Preliminaries

**Definition 2.1.** A *metric semigroup* is a structure $(X, \rho, +, \cdot)$, where $\rho : X \times X \to \mathbb{R}$, $+ : X \times X \to X$, $\cdot : \mathbb{R} \times X \to X$ satisfy the following conditions:

**(i)** $(X, \rho)$ is a complete metric space;

**(ii)** $(X, +)$ is a commutative semigroup endowed with a neutral element 0;

**(iii)** $\rho(w + y, z + t) \leq \rho(w, z) + \rho(y, t)$ for any $w, y, z, t \in X$;

**(iv)** $\rho(\alpha\, w, \alpha\, y) \leq |\alpha|\, \rho(w, y)$ for all $\alpha \in \mathbb{R}$ and $w, y \in X$;

**(v)** $\alpha(w + y) = \alpha w + \alpha y$ for each $\alpha \in \mathbb{R}$, $w, y \in X$;

**(vi)** $(\alpha + \beta)w = \alpha w + \beta w$ for every $\alpha, \beta \in \mathbb{R}_0^+$, $w \in X$, $0 \cdot w = 0$ and $1 \cdot w = w$ for each $w \in X$.

A metric semigroup $(X, \rho, +, \cdot)$ is said to be *invariant,* if

$$\rho(w + z, y + z) = \rho(w, y)$$

for any $w, y, z \in X$.

An example of metric semigroup, which is not a group, is the set of all fuzzy numbers (see also [1, 5]).

For what concerns the Kurzweil-Henstock integral and its properties for functions with values in a metric semigroup $X$ we refer to [5, § 3]. For the sake of simplicity we recall here only the main definitions and the Henstock Lemma.

From now on, let $B$ be a connected subset of the extended real line and denote with $a < b \in \widetilde{\mathbb{R}}$ its endpoints. Moreover, for every measurable set $E \subset \widetilde{\mathbb{R}}(\widetilde{\mathbb{R}}^2)$, denote by $|E|$ or $\lambda_1(E)(\lambda_2(E))$ its Lebesgue one- (two-)dimensional measure.

A *gauge* on $[a,b]$ is a map $\delta$ which associates to every point of $[a,b]$ an open subset of $\widetilde{\mathbb{R}}$, such that $x \in \delta(x)$ for all $x \in [a,b]$ and $\delta(x)$ is bounded whenever $x \in [a,b] \cap \mathbb{R}$.

Given a gauge $\delta$ on $[a,b]$, a partition $\Pi = \{(I_k, t_k), k = 1, \dots, p\}$ of $[a,b]$ is said to be *$\delta$-fine* if $t_k \in I_k \subset \delta(t_k)$, $k = 1, \dots, p$.

**Definition 2.2.** ([5, Definition 3.2]) We say that a function $f : [a,b] \to X$ is *Kurzweil-Henstock integrable* (in short *integrable* ) on $[a,b]$ if there exists an element $I \in X$ such that for all $\varepsilon > 0$ there is a gauge $\delta$ on $[a,b]$ with

$$\rho\left(I, \sum_{\substack{k=1,\dots,p \\ \lambda_1(I_k)<+\infty}} \lambda_1(I_k)\, f(t_k)\right) \leq \varepsilon \tag{1}$$

whenever $\Pi = \{(I_k, t_k), k = 1, \dots, p\}$ is a $\delta$-fine partition of $[a,b]$. In this case we say that $I$ is the *$KH$-integral of $f$,* we denote the element $I$ by the symbol $\displaystyle\int_a^b f$ and the sum in (1) (*Riemann sum*) by $\displaystyle\sum_{\Pi} f$.

**Lemma 2.3.** [5, Proposition 4.1] (Henstock Lemma) *Let $f : [a,b] \to X$ be integrable, $\varepsilon > 0$, and $\delta$ be a gauge on $[a,b]$ such that*

$$\rho\left(\sum_{\Pi} f, \int_a^b f\right) \leq \varepsilon$$

*whenever $\Pi$ is any $\delta$-fine partition of $[a,b]$. Let $A_i \subset [a,b]$, $i = 1, \dots, m$, be nonoverlapping intervals (one or two of them may be halflines) and $t_i \in A_i$ be such that*

$$A_i \subset \delta(t_i) \quad (i = 1, \dots, m).$$

*Then*

$$\rho\left(\sum_{\substack{i=1,\dots,m, \\ \lambda_1(A_i)<+\infty}} \lambda_1(A_i)\, f(t_i), \sum_{i=1}^{m} \int_{A_i} f\right) \leq \varepsilon.$$

# 3 The integral in the two-dimensional case

Proceeding analogously as in [1, 2], it is possible to define a Kurzweil-Henstock type integral for metric semigroup-valued functions, defined in a (possibly unbounded) closed subrectangle $J$ of $\widetilde{\mathbb{R}}^2$, $J = H \times K$. We denote by $\mathcal{C}$ the family of all closed subrectangles of $J$.

- A *gauge* on $J$ is a map $\delta$ defined on $J$ and taking values in the set of all open subsets of $\widetilde{\mathbb{R}}^2$, such that $\vec{t} \in \delta(\vec{t}\,)$ for every $\vec{t} \in J$ and $\delta(\vec{t}\,)$ is bounded whenever $\vec{t} \in \mathbb{R}^2$.
- A *partition* of $J$ is a finite collection $\Pi = \{(W_i, \vec{t}_i) : i = 1, \dots, q\}$ such that

**(i)** $\bigcup_{i=1}^{q} W_i = J$;

**(ii)** $\vec{t_i} \in W_i$, $W_i \in \mathcal{C}$;

**(iii)** $\lambda_2(W_i \cap W_j) = 0$ whenever $i \neq j$.

A collection $\Pi$ satisfying axioms **(ii)** and **(iii)**, but not necessarily **(i)**, is called *decomposition* of $J$. The partition or decomposition $\Pi$ is $\delta$*-fine* if $W_i \subset \delta(\vec{t_i})$ $(i = 1, 2, \ldots, q)$.

Observe that every gauge $\delta$ on $J$ has a $\delta$-fine partition (of $J$), see [7, Lemma 6.2.6].

**Definition 3.1.** Given any partition $\Pi = \{(W_i, \vec{t_i}), i = 1, \ldots, q\}$ of $J$ and a function $f : J \to R$, the *Riemann sum* of $f$ is

$$\sum_{\Pi} f := \sum_{\substack{i=1,\ldots,q \\ \lambda_2(W_i)<+\infty}} \lambda_2(W_i) f(\vec{t_i}),$$

We now formulate our definition of Kurzweil-Henstock integral for $X$-valued functions defined on a closed (not necessarily bounded) subrectangle $J \subset \widetilde{\mathbb{R}}^2$.

**Definition 3.2.** We say that a two-variable function $f : J \to R$ is *Kurzweil-Henstock integrable* (in short *integrable* ) on $J$ if there exists an element $I \in R$ such that for every $\varepsilon > 0$ there corresponds a gauge $\delta$ with

$$\rho\left(I, \sum_{\Pi} f\right) \leq \varepsilon \tag{2}$$

whenever $\Pi = \{(W_i, \vec{t_i}), i = 1, \ldots, q\}$ is a $\delta$-fine partition of $J$. In this case we say that $I$ is the *Kurzweil-Henstock integral of* $f$, and denote the element $I$ by the symbol $\int_J f$.

It is easy to check that the integral just defined is uniquely determined, linear and satisfies the Bolzano-Cauchy condition.

# 4 Convergence theorems

We begin with the two-dimensional version of the Henstock lemma (see also [5, Proposition 4.1]).

**Lemma 4.1.** *Let $f : J \to X$ be integrable, $\varepsilon > 0$, and $\delta$ be a gauge on $J$ such that*

$$\rho\left(\sum_{\Pi} f, \int_J f\right) \leq \varepsilon$$

*whenever $\Pi$ is any $\delta$-fine partition of $J$. Let $A_i \subset J$, $i = 1, \ldots, m$, be elements of $\mathcal{C}$, with $\lambda_2(A_i \cap A_j) = 0$ whenever $i \neq j$, and $\vec{t_i} \in A_i$ be such that*

$$A_i \subset \delta(\vec{t_i}) \quad (i = 1, \ldots, m).$$

*Then*

$$\rho\left(\sum_{\substack{i=1,\dots,m,\\ \lambda_2(A_i)<+\infty}} \lambda_2(A_i)f(\vec{t}_i), \sum_{i=1}^{m}\int_{A_i} f\right) \le \varepsilon.$$

**Proof:** Let $A_i^\circ$ be the interior of $A_i$, $i = 1,\dots,m$. Since the $A_i$'s are non-overlapping, the set $J \setminus \cup_{i=1}^{m} A_i^\circ$ is empty or it is the union of non-overlapping (possibly unbounded) rectangles $B_1,\dots,B_p$. Let $\eta > 0$. Since $f$ is integrable on each $B_j$, for each $j = 1,\dots,p$ there exists a gauge $\delta_j$ on $B_j$ such that

$$\delta_j(x) \subset \delta(x) \qquad \text{for all } \; x \in B_j,$$

and

$$\rho\left(\sum_{\Pi_j} f, \int_{B_j} f\right) < \frac{\eta}{p+1}$$

for every $\delta_j$-fine partition $\Pi_j$ of $B_j$. Let now $\Pi_j$ be any $\delta_j$-fine partition of $B_j$. We observe that

$$\Pi \equiv \{(A_i, \vec{t}_i), i = 1,\dots,m\} \cup (\cup_{j=1}^{p} \Pi_j)$$

is a $\delta$-fine partition of $J$. Then we have:

$$\begin{aligned}
& \rho\left(\sum_{\substack{i=1,\dots,m,\\ \lambda_2(A_i)<+\infty}} \lambda_2(A_i)\, f(\vec{t}_i), \sum_{i=1}^{m}\int_{A_i} f\right) \\
= \;& \rho\left(\sum_{\substack{i=1,\dots,m,\\ \lambda_2(A_i)<+\infty}} \lambda_2(A_i)\, f(\vec{t}_i) + \sum_{j=1}^{p}\sum_{\Pi_j} f,\; \sum_{i=1}^{m}\int_{A_i} f + \sum_{j=1}^{p}\sum_{\Pi_j} f\right) \\
\le \;& \rho\left(\sum_{\Pi} f, \int_J f\right) + \rho\left(\sum_{i=1}^{m}\int_{A_i} f + \sum_{j=1}^{p}\int_{B_j} f, \sum_{i=1}^{m}\int_{A_i} f + \sum_{j=1}^{p}\sum_{\Pi_j} f\right) \\
\le \;& \varepsilon + \rho\left(\sum_{j=1}^{p}\int_{B_j} f, \sum_{j=1}^{p}\sum_{\Pi_j} f\right) \le \varepsilon + \sum_{j=1}^{p}\rho\left(\int_{B_j} f, \sum_{\Pi_j} f\right) \\
< \;& \varepsilon + \sum_{j=1}^{p}\frac{\eta}{p+1} < \varepsilon + \eta.
\end{aligned}$$

Since the inequality

$$\rho\left(\sum_{\substack{i=1,\dots,m,\\ \lambda_2(A_i)<+\infty}} \lambda_2(A_i)f(\vec{t}_i), \sum_{i=1}^{m}\int_{A_i} f\right) < \varepsilon + \eta$$

holds for any $\eta > 0$, then we get the assertion. $\square$

**Definition 4.2.** A sequence of integrable functions $(f_k : J \to X)_k$ is said to be *equiintegrable* if to any $\varepsilon > 0$ there exists a gauge $\delta$ on $J$ such that

$$\rho\left(\sum_{\Pi} f_k, \int_J f_k\right) \leq \varepsilon$$

for any $\delta$-fine partition $\Pi$ and every $k \in \mathbb{N}$.

**Theorem 4.3.** *Let $(f_k : J \to X)_k$ be an equiintegrable sequence and let*

$$\lim_{k\to+\infty} \rho(f_k(\vec{t}), f(\vec{t})) = 0$$

*for any $\vec{t} \in J$. Then $f$ is integrable on $J$, and*

$$\lim_{k\to+\infty} \rho\left(\int_J f_k, \int_J f\right) = 0.$$

**Proof:** Fix $\varepsilon > 0$. There exist an integrable function $E : J \cap \mathbb{R}^2 \to \mathbb{R}^+$, with

$$\int_J E \leq \frac{\varepsilon}{2}$$

(for example, $E(\vec{t}\,) = \dfrac{\varepsilon}{16}\, e^{-|\vec{t}|^2}, \quad \vec{t} \in J \cap \mathbb{R}^2$), and a gauge $\delta_0$ on $J$, such that

$$\sum_{\substack{i=1,\ldots,n,\\ \lambda_2(I_i)<+\infty}} \lambda_2(I_i)\, E(\vec{t_i}) \leq \varepsilon$$

for each $\delta_0$-fine partition $\Pi$ of $J$, $\Pi = \{(I_i, \vec{t_i}), i = 1, \ldots, n\}$. Let now $\varepsilon > 0$, $\delta$ be as in Definition 4.2, $\widehat{\delta} = \delta \cap \delta_0$, and $\Pi$ be any $\widehat{\delta}$-fine partition of $J$. Then, definitely on $k$, we get

$$\rho\left(\sum_{\Pi} f_k, \sum_{\Pi} f\right) \leq \sum_{\substack{i=1,\ldots,n,\\ \lambda_2(I_i)<+\infty}} \lambda_2(I_i)\, E(\vec{t_i}) \leq \varepsilon.$$

So,

$$\lim_{k\to+\infty} \rho\left(\sum_{\Pi} f_k, \sum_{\Pi} f\right) = 0.$$

Thanks to equiintegrability, the sequence $\left(\int_J f_k\right)_k$ of elements of $X$ is Cauchy, and by completeness of $(X, \rho)$ there exist $I \in X$, a gauge $\delta^*$ on $J$ and an integer $k_0$ such that

$$\rho\left(I, \int_J f_k\right) \leq \varepsilon \quad \text{for any } k \geq k_0,$$

and for each $\delta^*$-fine partition $\Pi$ of $J$, there is $k_1 \geq k_0$, such that

$$\rho\left(\sum_{\Pi} f_k, \sum_{\Pi} f\right) \leq \varepsilon \quad \text{for any } k \geq k_1 \geq k_0.$$

Moreover, for each $\delta \cap \delta^*$-fine partition $\Pi$, we get:

$$\rho\left(I, \sum_{\Pi} f\right) \le \rho\left(\sum_{\Pi} f, \sum_{\Pi} f_{k_1}\right) + \rho\left(\sum_{\Pi} f_{k_1}, \int_J f_{k_1}\right) + \rho\left(I, \int_J f_{k_1}\right) \le 3\varepsilon.$$

So $f$ is integrable and $I = \displaystyle\int_J f$, and this is enough to prove the assertion, since the limit follows easily. $\square$

We now prove a Hake-type theorem.

**Theorem 4.4.** *Let $f : J \to X$, with $f(x, y) = 0$ whenever $x = \pm\infty$ or $y = \pm\infty$. Suppose that $f$ is integrable on each $A \in \mathcal{C}$, $A \ne J$, and that there is $I \in X$ satisfying the following condition:*

**(4.4.1)** *for all $\varepsilon > 0$ there corresponds a set $A^* \ne J$, with the properties that $A^*$ is a finite union of non-overlapping elements of $\mathcal{C}$ and*

$$\rho\left(I, \int_A f\right) \le \varepsilon$$

*whenever $J \ne A \supset A^*$ and $A$ is a finite union of non-overlapping sets of $\mathcal{C}$.*

*Then $f$ is integrable on $J$ and $\displaystyle\int_J f = I$.*

**Proof:** Let $(A_n)_n$ be a sequence in $\mathcal{C}$ with $A_n \subset A_{n+1}$ for any $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} A_n = J \cap \mathbb{R}^2$. Note that for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a gauge $\delta_n$ on $A_n$ with

$$\rho\left(\sum_{\Pi_n} f, \int_{A_n} f\right) \le \frac{\varepsilon}{2^n} \tag{3}$$

for any $\delta_n$-fine partition $\Pi_n$ of $A_n$.

Put $C_n = A_n \setminus A_{n-1}$, $A_0 = \emptyset$. For every $\vec{t} \in J \cap \mathbb{R}^2$ there exists exactly one positive integer $n = n(\vec{t})$ with $\vec{t} \in C_n$. Choose now a gauge $\delta$ on $J$ with the property that $\delta(\vec{t}) \subset \delta_{n(\vec{t})}(\vec{t})$ and $\delta(\vec{t}) \subset A^{\circ}_{n(\vec{t})}$ whenever $\vec{t} \in J \cap \mathbb{R}^2$, and $\delta(x, y) \subset J \setminus A^*$ whenever $x = \pm\infty$ or $y = \pm\infty$ (here, $A^*$ is the set in (4.4.1)). Let $\Pi := \{(\mathcal{U}_i, \vec{t}_i) : i = 1, \dots, q\}$ be a $\delta$-fine partition of $J$. For every $i = 1, \dots, q$ we get: $\mathcal{U}_i \subset \delta(\vec{t}_i) \subset A_{n(\vec{t}_i)}$, and $\mathcal{U}_i \subset \delta_{n(\vec{t}_i)}(\vec{t}_i)$.

If $J$ is unbounded, then there exists at least $(\mathcal{U}_{i_0}, \vec{t}_{i_0}) \in \Pi$, with $i_0 \in \{1, 2, \dots, q\}$, such that $\vec{t}_{i_0} =: (x_0, y_0) \in \mathcal{U}_{i_0}$ and $\lambda_2(\mathcal{U}_{i_0}) = +\infty$. Then $x_0 = \pm\infty$ or $y_0 = \pm\infty$. If not, then for some positive integer $n$

$$\vec{t}_{i_0} \in \mathcal{U}_{i_0} \subset \delta(\vec{t}_{i_0}) \subset \delta_n(\vec{t}_{i_0}) \subset A_{n(\vec{t}_{i_0})} \subset J \cap \mathbb{R}^2,$$

a contradiction.

Let $\mathcal{I}$ be the set of those indexes $i$ for which $\lambda_2(\mathcal{U}_i) < +\infty$. Let $A = \cup_{i \in \mathcal{I}} \mathcal{U}_i$: since $\Pi$ is a $\delta$-fine *partition* of $J$, by construction we get: $A \supset A^*$. Without loss of generality, we can suppose that $A \in \mathcal{C}$, since the case of finite unions of non-overlapping elements of $\mathcal{C}$ is obtained by additivity. By (4.4.1) we have

$$\rho\left(I, \int_A f\right) \le \varepsilon.$$

By the Henstock Lemma 4.1 we obtain

$$\rho\left(\sum_{\vec{t}_i \in C_n} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i), \sum_{\vec{t}_i \in C_n} \int_{\mathcal{U}_i} f\right) \le \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. By additivity, we get:

$$\int_A f = \sum_{i \in \mathcal{I}} \int_{\mathcal{U}_i} f,$$

and hence

$$\begin{aligned}
\rho\left(\sum_{i \in \mathcal{I}} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i), \int_A f\right) &= \rho\left(\sum_{i \in \mathcal{I}} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i), \sum_{i \in \mathcal{I}} \int_{\mathcal{U}_i} f\right) \\
&\le \sum_{n=1}^{\infty} \left\{\rho\left(\sum_{\vec{t}_i \in C_n} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i), \sum_{\vec{t}_i \in C_n} \int_{\mathcal{U}_i} f\right)\right\} \\
&\le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
\end{aligned}$$

Thus we get:

$$\begin{aligned}
\rho\left(I, \sum_{\Pi} f\right) &= \rho\left(I, \sum_{i \in \mathcal{I}} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i)\right) \\
&\le \rho\left(\sum_{i \in \mathcal{I}} \lambda_2(\mathcal{U}_i)\, f(\vec{t}_i), \int_A f\right) + \rho\left(I, \int_A f\right) \le 2\,\varepsilon.
\end{aligned}$$

Thus the assertion follows. □

# 5 More properties of the integral and a Fubini Theorem

Similarly as [5, Proposition 3.12] and [14, Lemma 7], it is possible to prove the following properties:

**Lemma 5.1.** *Let $f : J \to X$ be integrable on $J$ and $g : J \to X$ with $f = g$ in the complement of a $\lambda_2$-null set $W \subset J$. Then $g$ is integrable on $J$ too and the two integrals coincide.*

**Proof:** Let $I = \int_J f$ and fix $\varepsilon > 0$. Then a gauge $\overline{\delta}$ on $J$ can be found, with

$$\rho\left(I, \sum_{\Pi} f\right) \leq \frac{\varepsilon}{2}$$

whenever $\Pi$ is a $\overline{\delta}$-fine partition of $J$. Set

$$W_l = \{\vec{t} \in J : \rho(f(\vec{t}), g(\vec{t})) \in (l-1, l]\},\ l \in \mathbb{N}; \quad W = \cup_{l=1}^{\infty} W_l.$$

For each $l$, as $\lambda(W_l) = 0$, there exists a set $G_l \supset W_l$ which is union of countably many open intervals with total length less than $\dfrac{\varepsilon}{2^{l+1}\, l}$. Put

$$\delta(\vec{t}) = \begin{cases} \overline{\delta}(\vec{t}), \text{ if } \vec{t} \notin W; \\ G_l, \text{ if } \vec{t} \in W_l. \end{cases}$$

Choose an arbitrary partition $\Pi$ of $J$, $\Pi = \{(L_i, \vec{t}_i), i = 1, \ldots, n\}$. We have:

$$\rho\left(I, \sum_{i=1}^{n} \lambda_2(L_i)\, g(\vec{t}_i)\right) = \rho\left(I, \sum_{\vec{t}_i \notin W} \lambda_2(L_i) g(\vec{t}_i) + \sum_{\vec{t}_i \in W} \lambda_2(L_i) g(\vec{t}_i)\right) =$$

$$\rho\left(\sum_{\vec{t}_i \notin W} \lambda_2(L_i) f(\vec{t}_i) + \sum_{\vec{t}_i \in W} \lambda_2(L_i) g(\vec{t}_i) + \sum_{\vec{t}_i \in W} \lambda_2(L_i) f(\vec{t}_i), I + \sum_{\vec{t}_i \in W} \lambda_2(L_i) f(\vec{t}_i)\right) =$$

$$\rho\left(\sum_{i=1}^{n} \lambda_2(L_i) f(\vec{t}_i) + \sum_{\vec{t}_i \in W} \lambda_2(L_i) g(\vec{t}_i),\ I + \sum_{\vec{t}_i \in W} \lambda_2(L_i) f(\vec{t}_i)\right) \leq$$

$$\rho\left(I, \sum_{\Pi} f\right) + \sum_{\vec{t}_i \in W} \lambda_2(L_i)\, \rho(g(\vec{t}_i), f(\vec{t}_i)) \leq \frac{\varepsilon}{2} + \sum_{l=1}^{\infty} \sum_{\vec{t}_i \in W_l} \lambda_2(L_i)\, \rho(g(\vec{t}_i), f(\vec{t}_i)) \leq$$

$$\frac{\varepsilon}{2} + \sum_{l=1}^{\infty} l \sum_{\vec{t}_i \in W_l} \lambda_2(L_i) \leq \frac{\varepsilon}{2} + \sum_{l=1}^{\infty} l \frac{\varepsilon}{2^{l+1}\, l} = \varepsilon.$$

From this it follows that $g$ is integrable on $J$ and $\int_J g = I$. This concludes the proof. □

Moreover we have also that:

**Lemma 5.2.** *Let $f : J \to X$ be integrable on $J = H \times K$, and set $N := \{y \in K : f(\cdot, y)$ is not integrable on $H\}$. Then $N$ is null, that is $\int_K 1_N = 0$.*

**Proof:** Let $(R_n)_n$ be a sequence of pairwise nonoverlapping closed bounded rectangles, $R_n := [a_n, b_n] \times [c_n, d_n]$, $n \in \mathbb{N}$, such that $\cup_{n=1}^{\infty} R_n = J \cap \mathbb{R}^2$.

We know that $f$ is integrable on each rectangle $R_n$. From [14, Lemma 5] it follows that, for each $n \in \mathbb{N}$, the set $N_n := \{y \in [c_n, d_n] : f(\cdot, y)$ is not integrable on $[a_n, b_n]\}$ is null, that is $\int_{c_n}^{d_n} 1_{N_n} = 0$ (see also [7]). Since every countable union of null sets is still a null set, the set $N := \{y \in K : f(\cdot, y)$ is not integrable on $H\}$ is null, and from this we get the assertion. $\square$

We are now able to give our version of the Fubini Theorem, thanks to Lemmata 5.1 and 5.2.

**Theorem 5.3.** (Fubini Theorem) *Let $f : J = H \times K \to R$ be integrable and set*

$$Q(x) = \int_K f(x, y)\, dy.$$

*Then, $Q$ is integrable for (almost all) $x \in H$ and*

$$\int_H \left( \int_K f(x, y) dy \right) dx = \iint_{H \times K} f.$$

**Proof:** Observe that, by Lemma 5.2, $Q$ is integrable for $x \in H \setminus N$, where $\lambda_1(N) = 0$. Without loss of generality, we can define $Q(x) = 0$ for each $x \in N$ and so, thanks to Lemma 5.1, we can assume that $Q$ is integrable on the whole of $H$. Put $M = N \times K$. Then $\lambda_2(M) = 0$.

Fix arbitrarily $\varepsilon > 0$. By the Henstock Lemma there is a gauge $\delta$ on $J$ with

$$\rho \left( \sum_{l=1}^{q} \lambda_2(Z_l)\, f(\vec{t_l}), \sum_{l=1}^{q} \int_{Z_l} f \right) \leq \varepsilon \tag{4}$$

whenever $\Pi = \{(Z_l, \vec{t_l}) : l = 1, \ldots, q\}$ is a $\delta$-fine partition of $J$.

Let now $h^*$ be a positive real-valued integrable map, defined on $H$. There exist a gauge $\Delta_0$ on $H$ and a positive constant $C_0$ with the property that

$$\sum_{\pi_0} h^* = \sum_{\pi_0} |V_j| h^*(\eta_j) \leq C_0$$

for every $\Delta_0$-fine partition $\pi_0$ of $H$, $\pi_0 = \{(V_j, \eta_j) : j = 1, \ldots, s\}$.

Set now $\vec{t} = (x, y)$. Given a gauge $\delta(x, y) = (U_1(x, y) \times U_2(x, y))$, for any fixed $x \in H$ the mapping $\delta_{K,x}(y) = U_2(x, y)$ is a gauge on $K$. Analogously, for every $y \in K$, the map $\delta_{H,y}(x) = U_1(x, y)$ is a gauge on $H$. These two mappings are the *projections* of $\delta$.

As in [7, Theorem 6.6.3], for $x \in H \setminus N$ let $\tau(x) = \{(K_i(x), y_i(x)) : i = 1, \ldots, n(x)\}$ be a $\delta_{K,x}$-fine partition for which

$$\rho \left( Q(x), \sum_{i=1}^{n(x)} f(x, y_i(x))\, |K_i(x)| \right) \leq h^*(x);$$

and for any $x \in N$ let $\phi(x)$ be a $\delta_{K,x}$-fine partition. Set $\sigma(x) = \tau(x)$ whenever $x \in H \setminus N$ and $\sigma(x) = \phi(x)$ if $x \in N$.

Let $\Delta$ be a gauge on $H$, according with [7, Lemma 6.6.2], with the property that, whenever $\pi_H = \{(H_j, z_j) : j = 1, \ldots, m\}$ is a $\Delta$-fine partition of $H$, then the associated compound partition

$$\{(H_j \times K_i(z_j), (z_j, y_i(z_j))) : j = 1, \ldots, m, \, i = 1, \ldots, n(z_j)\} \tag{5}$$

is $\delta$-fine.

Without loss of generality, we can choose $\Delta$ such that $\Delta(x) \subset \Delta_0(x)$ for every $x \in H$. Pick a $\Delta$-fine partition $\pi_H = \{(H_j, z_j) : j = 1, \ldots, m\}$ of $H$, and let

$$\pi = \{(H_j \times K_i(z_j), (z_j, y_i(z_j))) : j = 1, \ldots, m, \, i = 1, \ldots, n(z_j)\}$$

be as in (5). We have:

$$\begin{aligned}
\rho\left(I, \sum_{\pi_H} Q(z_j)|H_j|\right) &\leq \rho\left(\sum_{\pi_H} |H_j| Q(z_j), \sum_{\pi_H} \sum_{\sigma(z_j)} f(z_j, y_i(z_j))\, |H_j|\, |K_i(z_j)|\right) \\
&+ \rho\left(\sum_{\pi} f(z_j, y_i(z_j))\, |H_j|\, |K_i(z_j)|, \sum_{\pi} \int_{H_j \times K_i(z_j)} f\right) \\
&\leq \sum_{\pi_H} |H_j| \rho\left(Q(z_j), \sum_{\sigma(z_j)} f(z_j, y_i(z_j)) |K_i(z_j)|\right) \\
&+ \rho\left(\sum_{\pi} f(z_j, y_i(z_j))\, |H_j|\, |K_i(z_j)|, \sum_{\pi} \int_{H_j \times K_i(z_j)} f\right) \\
&\leq \varepsilon \sum_{\pi_H} |H_j| h^*(z_j) + \varepsilon \leq C_0\, \varepsilon + \varepsilon.
\end{aligned}$$

This concludes the proof. $\square$

**Remark 5.4.** The function $Q$ represents the integral of the "$x$-projections" with respect to the variable $y$. The Lemma 5.2 allows us to say that, in the context of metric semigroups, the map $Q$ is well-defined in the complement of a $\lambda_1$-null sets. Similar results ([2, Theorem 4.3]) were given in the context of Riesz spaces.

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