Zeitschrift für Analysis ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 4, 1-14

The finitely additive integral of multifunctions with closed and convex values

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Abstract: We investigate integration with respect to a finitely additive measure of integrands with closed, convex values and we obtain a closedness result for the Aumann integral.

Key words: *Pettis integral, Aumann integral, Debreu integral, Stone extension, integrable selections, finitely additive measures.*
Mathematical Subject Classification: 28B20, 26E25, 54C60, 54C65, 46G10

1 Introduction

The importance for Mathematical Economics of integrating correspondences with values in infinite dimensional commodity spaces is widely recognized. This has lead to an increasing interest in multivalued integration of Banach-valued integrands. However the finitely additive case, that would be of great interest for defining perfect competitions, according to [3], has been quite neglected so far.

In [20] we investigated integration with respect to a finitely additive measure of integrands with compact and convex values, fulfilling suitable measurability assumptions, and we obtained a compactness and convexity result for the Aumann integral.

In this paper we consider integrands that are only weakly compact and convex valued. The range space is assumed to be a separable and reflexive Banach space.

The first problem arising in the finitely additive setting is the existence of selections; the condition assumed in [20] to overcome this difficulty immediately leads to the compactness of the values of the integrand, so it can't be used here. Therefore here we define the Aumann integral by means of the Pettis integrable selections.

The Aumann integral, obtained via Pettis integrable selections, was already studied in the countably additive case in [1, 2] and [25, 26] and it is also used in the paper of Tourki and Yannelis [24] in order to compare the core and the set of walrasian Pettis integrable allocations in a suitable economy. The allocation they obtain exhibits a "pathological behaviour" of the Pettis integral, namely

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This work is partiaqlly supported by G.N.AM.P.A.

ISSN 0232-2064 / ©

integrands with values in the positive cone of a non separable Banach lattice may give a zero integral. This is not the case here due to the separability of the target space.

For the Aumann integral we obtain, under standard measurability assumptions, an extension of the classical theorem on existence of measurable selections holding in a measure space ([8] III.6, page 65). Then we define scalarly the Stone extensions for the selections, using the identification of the target space with its bidual.

Finally, with the use of Radström's Embedding Theorem and the identification of the Aumann and the Debreu integral for the Stone extension we achieve the closedness and the convexity of the Aumann integral in this more general case.

We would like to thank Domenico Candeloro for his advice, suggestions and kind help. We are also greatly indebted to Achille Basile who suggested us many improvements as well as to Hans Weber, Patrizia Berti and Pietro Rigo who pointed out to us some examples.

2 Definitions and notation

Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\mu : \Sigma \to [0, +\infty)$ a bounded finitely additive measure. Further, let X be a reflexive, separable Banach space. With X^* we denote its topological dual and with X_1 and X_1^* the unit balls of X and X^*, respectively.

Definition 2.1 A function $f : \Omega \to X$ is said to be

(i) Σ-measurable if $f^{-1}(B) \in \Sigma$ for every Borelian set B of X;

(ii) totally measurable if there exists a sequence of simple functions $(f_n)_n$ μ-converging to f;

(iii) scalarly μ-measurable if x^*f is totally measurable for all $x^* \in X^*$.

Note that the condition of Σ-measurability of f in turn implies that x^*f has measurable level sets, for every $x^* \in X^*$. We denote by $L^1_\mu(X)$ the space of totally measurable functions f which are Bochner integrable [13, Definition III.2.17]. Observe that, if f is Σ-measurable and $|f|$ is dominated by some $g \in L^1_\mu(\mathbb{R}^+_0)$, then f is scalarly μ-measurable.

We now introduce the Pettis integral in the finitely additive case.

Definition 2.2 Let f be a scalarly μ-measurable function such that $x^*f \in L^1_\mu(\mathbb{R})$ for all $x^* \in X^*$. If, for all $E \in \Sigma$, there exists $x_E \in X$ such that $x^*(x_E) = \displaystyle\int_E x^* f d\mu$ then f is said to be *Pettis integrable.*

We denote by $P(\mu, X)$ the space of all Pettis integrable functions $f : \Omega \to X$ and we introduce in this space the usual norm

$$|f|_P = \sup_{x^* \in X_1^*} \int_\Omega |x^* f| d\mu.$$

Further, we denote by $cb(X)$, (resp. $ck(X)$) the family of non empty, bounded, convex, closed (resp. compact) subsets of X. Let h be the Hausdorff distance on $cb(X)$, namely $h(C, D) =$

$\sup\{e(C,D), e(D,C)\}$, where $e(C,D) = \sup\{d(x,D), x \in C\}$ with $d(x,D) = \inf\{\|x-y\|, y \in D\}$. For all $C \in cb(X)$, we set $|C| = h(C,\{0\})$. In this paper we always consider multifunctions F with values in $cb(X)$.

Definition 2.3 A multifunction $F : \Omega \to cb(X)$ is said to be

(i) *strongly measurable* if $F^{-}(C) = \{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \Sigma$ for every closed set C;

(ii) *Effros measurable* or *weakly measurable* if $F^{-}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \Sigma$ for every open set A.

Obviously if F is strongly measurable it is also weakly measurable (since every open set is a $\mathcal{F}_\sigma$), while the converse fails to be true, (for an example we refer to [22, pag. 255]).

A multifunction F admits a *Castaing representation* if there exists a sequence of Σ-measurable scalarly μ-measurable selections $(\sigma_n)_n$ such that, for all $\omega \in \Omega$,

$$F(\omega) = cl\{\sigma_1(\omega), \sigma_2(\omega), \ldots\},$$

where clA denotes the closure of A. It is well known that, in the countably additive case, F is weakly measurable if and only if F admits a Castaing representation (when X is a separable complete metric space, and F takes closed, non empty values, see for example [8, Theorems III.6 and III.7]). In Corollary 3.5 we will give an analogous result in the finitely additive case.

We denote by S^1_F and $S^1_{F,P}$ the families of all the selections of F which are Bochner and Pettis integrable, respectively.

We can define the Aumann integral in this way:

Definition 2.4 If F is a (weakly) measurable multifunction such that $S^1_{F,P} \neq \emptyset$ then

$$(A) - \int_E F d\mu = \left\{ \int_E f d\mu : f \in S^1_{F,P} \right\}. \tag{1}$$

Definition 2.5 F is said to be *integrably bounded* if there exists a function $g \in L^1_\mu(I\!R^+_0)$ such that $|F|(\omega) \leq g(\omega)$, μ-almost everywhere.

We recall that if μ is countably additive, and F is integrably bounded, since X is separable, $S^1_F = S^1_{F,P}$ thanks to Pettis Theorem. (See [11], pag 42). So in the countably additive case we obtain the classical Aumann integral.

Remark 2.6 The Aumann integral defined by (1) is convex: in fact, if $f_1, f_2 \in S^1_{F,P}$ then, for all $\alpha \in [0,1]$, we have $\alpha f_1 + (1-\alpha) f_2 \in S^1_{F,P}$; moreover if F is integrably bounded it is also bounded:

$$\left|(A) - \int_E F d\mu\right| = \sup_{f \in S^1_{F,P}} \left\|(P) - \int_E f d\mu\right\| = \sup_{f \in S^1_{F,P}} \sup_{x^* \in X^*_1,} \left|\int_E x^* f d\mu\right| \leq \int_E g d\mu.$$

3 Existence of selections

Selection theorems have been given by several authors (see, for example, [8]). We give here an existence theorem for Pettis integrable selections in the finitely additive case.

Proposition 3.1 *If $F : \Omega \to cb(X)$ is weakly measurable and integrably bounded, then $S^1_{F,P} \neq \emptyset$.*

Proof: Thanks to the separability of X it is possible to construct, analogously to [8, Theorem III.6], a sequence of Σ-measurable functions $(f_p)_p$ which is uniformly Cauchy and such that $d(f_p(\omega), F(\omega)) = \inf\{\|f_p(\omega) - x\|, x \in F(\omega)\} < \dfrac{1}{2^p}$ for all $\omega \in \Omega$. Hence $f = \lim_n f_n$ exists. Also f is Σ-measurable and it is such that $f(\omega) \in F(\omega)$ since F has closed values.
Since F is integrably bounded, there exists $g \in L^1_\mu(I\!R^+_0)$ dominating F, and this implies the scalar μ-measurability and the scalar μ-integrability of f.

For every $E \in \Sigma$, let $T_E : X^* \to I\!R^+_0$ be the linear functional defined by $T_E(x^*) = \int_E x^* f \; d\mu$. Since g dominates F, we get

$$\sup_{x^* \in X^*_1} |T_E(x^*)| \quad \leq \quad \sup_{x^* \in X^*_1} \int_E |x^* f| d\mu \leq \int_\Omega g d\mu < \infty.$$

So T_E is continuous, i.e. $T_E \in X^{**}$. Since X is reflexive, $T_E \in X$; then f is Pettis integrable. In this way we have proved that $S^1_{F,P} \neq \emptyset$.

Remark 3.2 In the proof of Proposition 3.1, we can observe that the mere weak measurability ensures the existence of Σ-measurable selections, while to obtain the existence of Pettis integrable selections we also need that F is integrably bounded.

In [22] the following result is proven:

Proposition 3.3 ([22, Proposition 11.5.5]) *Let X be a separable metric space. Let $F, G : \Omega \to ck(X)$ be strongly measurable multifunctions. Then the set*

$$\Omega_0 = \{\omega \in \Omega : F(\omega) \cap G(\omega) \neq \emptyset\}$$

is strongly measurable and the multifunction $F \cap G : \Omega_0 \to ck(X)$ is strongly measurable.

However, one can easily show that the assumption "X separable metric space" can be replaced by "X is second countable". Therefore, since in our setting X is separable and reflexive the following result holds:

Proposition 3.4 *Let $F, G : \Omega \to 2^X \setminus \{\emptyset\}$ be two weakly measurable multifunctions, such that*

$$\Omega_0 = \{\omega \in \Omega : F(\omega) \cap G(\omega) \neq \emptyset\} \in \Sigma.$$

Then $I : \Omega_0 \to cb(X)$ defined by: $I(\omega) = F(\omega) \cap G(\omega)$ is weakly measurable.

Corollary 3.5 *$F : \Omega \to cb(X)$ is weakly measurable if and only if it admits a Castaing representation.*

Proof: is analogous to that of of [8, Theorem III.7] and is therefore omitted here.

Analogously to [8, Theorem III.41] the following statements can be proved:

Proposition 3.6 *Let $f : \Omega \to X$ be a Σ-measurable function, $F : \Omega \to cb(X)$ a weakly measurable and integrably bounded multifunction and $r : \Omega \to I\!R_0^+$ the scalar function defined by: $r(\omega) = \inf\{\|f(\omega) - x\|, x \in F(\omega)\}$. Then r is Σ-measurable.*

Proposition 3.7 *Let $f : \Omega \to X$ be a Σ-measurable function, $F : \Omega \to cb(X)$ a weakly measurable and integrably bounded multifunction and $r : \Omega \to I\!R_0^+$ the scalar function defined by: $r(\omega) = \inf\{\|f(\omega) - x\|, x \in F(\omega)\}$. Then for all $\omega \in \Omega$ there exists $x_\omega \in F(\omega)$ for which $r(\omega) = \|f(\omega) - x_\omega\|$.*

We introduce now the multifunction

$$\begin{aligned} \Gamma_f(\omega) &= \{x \in F(\omega) : \|f(\omega) - x\| = d(f(\omega), F(\omega))\} \\ &= F(\omega) \cap (f(\omega) + r(\omega)X_1)\,. \end{aligned}$$

by Proposition 3.7, Γ_f has non empty values for every $\omega \in \Omega$.

Proposition 3.8 *Let $F : \Omega \to cb(X)$ be a weakly measurable, integrably bounded multifunction, f be a Σ-measurable function and $r : \Omega \to I\!R_0^+$ the scalar function defined by: $r(\omega) = \inf\{\|f(\omega) - x\|, x \in F(\omega)\}$. Then the map Γ_f is weakly measurable and integrably bounded.*

Proof: By Proposition 3.6, r is Σ-measurable. The multifunction $B(\omega) = f(\omega) + r(\omega)X_1$ is then weakly measurable. In fact, if $\{u_n, n \in I\!N\}$ is dense in X_1, then $(\sigma_n)_n$ is a Castaing representation of B, where $\sigma_n(\omega) = f(\omega) + r(\omega)u_n$ and every σ_n is Σ-measurable. B is then weakly measurable by Corollary 3.5. By Proposition 3.4, Γ_f is weakly measurable and integrably bounded (since F is integrably bounded).

4 The Stone Extensions

Let $(S, \mathcal{G})$ be the Stone space associated to (Ω, Σ, μ), $\tau : \Sigma \to \mathcal{G}$ the Stone isomorphism, $\mathcal{G}_\sigma$ the σ-algebra generated by $\mathcal{G}$, and $\overline{\mu} : \mathcal{G}_\sigma \to I\!R_0^+$ the extended measure in the Stone sense of μ. The space $(S, \mathcal{G}_\sigma, \overline{\mu})$ is said to be the *Stone space* associated to (Ω, Σ, μ). For what concerns the terminology and results on this topic we refer to [23].

Definition 4.1 Let $F : \Omega \to cb(X)$ be a simple, weakly measurable multifunction,

$$F = \sum_{i=1}^{n} C_i 1_{E_i}$$

where $C_i \in cb(X)$ and $E_i \in \Sigma$ for all $i = 1, \cdots n$. We define $\overline{F} : S \to cb(X)$ in the following way:

$$\overline{F} = \sum_{i=1}^{n} C_i 1_{\tau(E_i)}.$$

Using Radström's Embedding Theorem [21, Theorem 2], the multifunction F can be viewed as a single valued function in a suitable Banach space B.

Definition 4.2 A weakly measurable multifunction $F : \Omega \to cb(X)$ is *totally measurable* if there exists a sequence of simple and weakly measurable multifunctions $(F_n)_n$ with values in $cb(X)$ such that for all $\alpha > 0$

$$\lim_{n\to\infty} \mu(\{\omega \in \Omega : h(F_n(\omega), F(\omega)) > \alpha\}) = 0,$$

where h is the Hausdorff distance.

We denote by $\mathcal{TM}(\Omega, \Sigma, \mu, cb(X))$ the space of all $cb(X)$-valued multifunctions that are totally measurable.

Definition 4.3 [9] A totally measurable multifunction $F : \Omega \to cb(X)$ is said to be *Debreu-integrable* (briefly (D)-integrable) if there exists a sequence of simple, weakly measurable multifunctions $(F_n)_n$ with values in $cb(X)$, such that

(i) $h(F_n, F)$ μ-converges to zero;

(ii) $\lim_{k,n\to\infty} \int_\Omega h(F_k, F_n) d\mu = 0$.

Then, for all $E \in \Sigma$, the sequence $\left((D) - \int_E F_n dm\right)_n$ is Cauchy in $(cb(X), h)$ and therefore it admits a limit in $(cb(X), h)$. We set then

$$(D) - \int_E F d\mu := \lim_{n\to\infty} (D) - \int_E F_n d\mu.$$

The sequence $(F_n)_n$ will be said *defining* for F. We denote by $L^1(\Omega, \Sigma, \mu, cb(X))$ the space of all $cb(X)$-valued multifunctions that are (D)-integrable.

If $F \in \mathcal{TM}(\Omega, \Sigma, \mu, cb(X))$ it is possible to construct its Stone extension $\overline{F}$ via simple multifunctions.

By the completeness of X, $\overline{F}$ takes values in $cb(X)$ $\overline{\mu}$-almost everywhere, and if F is integrably bounded, then $\overline{F}$ also is integrably bounded: in fact, let $\widetilde{g} \in L^1_{\overline{\mu}}(I\!R_0^+)$ be the Stone extension of $g = |F|$ that is $\widetilde{g} = \overline{|F|}$; it is known that $\widetilde{g} = |\overline{F}|$. Moreover, if $F \in L^1(\Omega, \Sigma, \mu, cb(X))$, then:

$$\begin{aligned}
(D) - \int_E F d\mu &= \lim_{n\to\infty} (D) - \int_E F_n d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} C_i^n \mu(E \cap E_i^n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{k_n} C_i^n \overline{\mu}(\tau(E) \cap \tau(E_i^n))
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} (D) - \int_{\tau(E)} \overline{F}_n d\overline{\mu} \\
&= (D) - \int_{\tau(E)} \overline{F} d\overline{\mu}.
\end{aligned}
$$

Our next scope is to define a Stone extension for a scalarly μ-measurable selection of an integrably bounded and totally measurable multifunction F. We suppose that Σ is complete with respect to μ, that is Σ contains all the subsets of the μ-null sets. In this case a scalarly μ-measurable selection of F is also scalarly Σ-measurable.

Let S_1 be the subset of S such that $\overline{|F|}(s)$ is real valued, for $s \in S_1$. Note that, since $\overline{|F|} \in L^1_{\overline{\mu}}(I\!R^+_0)$, then $\overline{\mu}(S_1^c) = \overline{\mu}(\{s \in S : \overline{|F|}(s) = +\infty\}) = 0$. Let $f : \Omega \to X$ be a scalarly μ-measurable function dominated by $|F|$. We will introduce a new function $\phi : S_1 \to X$ that will be the *Stone extension* of f.
Let $x^* \in X^*$ be fixed. The function x^*f is real valued and we denote by $\overline{x^*f}$ its Stone extension. For $s \in S_1$, let $\phi(s) : X^* \to I\!R$ be the functional defined by: $\phi(s)(x^*) = \overline{x^*f}(s)$. $\phi(s)$ is linear and bounded thanks to the properties of scalar Stone extension (see for example [23]). Therefore ϕ is defined $\overline{\mu}$-almost everywhere. We observe that, in effect, if f is totally measurable the extension obtained in this way coincides $\overline{\mu}$-a.e. with that obtained via simple functions.

Proposition 4.4 *The function ϕ is totally measurable (and (B)-integrable).*

Proof: On S_1, we have $x^*\phi = \overline{x^*f}$ by definition. Thus ϕ is scalarly measurable and so, thanks to the Pettis Theorem, totally measurable since X is separable and $\overline{\mu}$ is a measure. Since ϕ is dominated by $\overline{|F|} \in L^1_{\overline{\mu}}(I\!R^+_0)$, ϕ is also Bochner integrable.

5 Comparison of Aumann and Debreu integrals

From now on we suppose that Σ is complete with respect to μ. We recall that $\overline{F}$ is $\overline{\mu}$-almost everywhere $cb(X)$ valued. Let S_2 be the subset of all $s \in S_1$ such that $\overline{F}(s)$ is $cb(X)$ valued.

Theorem 5.1 *Let $F : \Omega \to cb(X)$ be a totally measurable and integrably bounded multifunction. If $f \in S^1_{F,P}$, then its Stone extension belongs to $S^1_{\overline{F}}$.*

Proof: By Proposition 4.4, $\phi \in L^1_{\overline{\mu}}(X)$. We only have to prove that ϕ is a selection of $\overline{F}$.
The multifunction $F_{x^*} : \Omega \to ck(I\!R)$ defined by

$$F_{x^*}(\omega) = \{x^*(x) : x \in F(\omega)\} = x^*(F(\omega))$$

is compact, convex valued. We now prove that $\overline{F}_{x^*} = \overline{F_{x^*}}$ $\overline{\mu}$-almost everywhere. If F is simple, $F = \sum_{i=1}^n C_i 1_{E_i}$ then by definition $\overline{F} = \sum_{i=1}^n C_i 1_{\tau(E_i)}$ and so

$$
\begin{aligned}
\overline{F}_{x^*} &= \sum_{i=1}^n x^*(C_i) 1_{\tau(E_i)}, \\
\overline{F_{x^*}} &= \overline{\sum_{i=1}^n x^*(C_i) 1_{E_i}} = \sum_{i=1}^n x^*(C_i) 1_{\tau(E_i)}.
\end{aligned}
$$

Consequently, the two multifunctions coincide in the simple case.
We suppose now that F is totally measurable. Let $(F_n)_n$ be a defining sequence for F. We prove that $((F_n)_{x^*})_n$ is defining for F_{x^*}. Indeed

$$\begin{aligned} e\left((F_n)_{x^*}(\omega), F_{x^*}(\omega)\right) &= \sup_{t\in x^*F_n(\omega)} d(t, F_{x^*}(\omega)) \\ &= \sup_{t\in x^*F_n(\omega)} \inf_{s\in F_{x^*}(\omega)} |t-s| \\ &= \sup_{\eta\in F_n(\omega)} \inf_{\sigma\in F(\omega)} |x^*\eta - x^*\sigma| \\ &\leq \sup_{\eta\in F_n(\omega)} \inf_{\sigma\in F(\omega)} \|\eta-\sigma\| \\ &= e(F_n(\omega), F(\omega)). \end{aligned}$$

The same relation holds for the other excess, so

$$h_{I\!R_0^+}((F_n)_{x^*}(\omega), F_{x^*}(\omega)) \leq h_X(F_n(\omega), F(\omega)).$$

Then, by definition of Stone extension, $\overline{(F_n)_{x^*}}$ $\overline{\mu}$-converges to $\overline{F_{x^*}}$. Since $\overline{(F_n)_{x^*}} = (\overline{F}_n)_{x^*}$ for all $n \in I\!N$, and $\overline{F}_n$ is defining for $\overline{F}$, then $(\overline{F}_n)_{x^*}$ is defining for $\overline{F}_{x^*}$ and so $\overline{F}_{x^*} = \overline{F_{x^*}}$ $\overline{\mu}$-almost everywhere. (Obviously the $\overline{\mu}$-null set depends on x^*.) So we have that for all $x^* \in X^*$ there exists a $\overline{\mu}$-null set N_{x^*} such that $x^*\phi \in \overline{F_{x^*}}$ for all $\omega \notin N_{x^*}$. In fact, $x^*f \in S^1_{F_{x^*}}$ for all $x^* \in X^*$, and so by [20, Theorem 5.1] $x^*\phi = \overline{x^*f} \in S^1_{\overline{F_{x^*}}}$. Since $\overline{F_{x^*}} = \overline{F}_{x^*}$ $\overline{\mu}$-almost everywhere we have obtained that $x^*\phi \in S^1_{\overline{F}_{x^*}}$, i.e. $x^*\phi(s) \in \overline{F}_{x^*}(s)$ for all $s \notin N_{x^*}$.
Let $D = \{x^*_n : n \in I\!N\}$ be dense in X^*, and set $\widetilde{S} = S_2 \setminus \cup_n N_{x^*_n}$. By the countable additivity of $\overline{\mu}$, we have $\overline{\mu}(\widetilde{S}) = \overline{\mu}(S_2)$. Moreover, if we suppose that there exists $s \in \widetilde{S}$, such that $\phi(s) \notin \overline{F}(s)$, a separation argument ([13], n.10, pag. 417) leads to a contradiction.

Proposition 5.2 *For every $E \in \Sigma$ and for every $f \in S^1_{F,P}$ one has*

$$(P)-\int_E f d\mu = (B)-\int_{\tau(E)} \phi d\overline{\mu}.$$

Proof: It is enough to observe that, for every $x^* \in X^*$ fixed,

$$\begin{aligned} x^*\left((B)-\int_{\tau(E)} \phi d\overline{\mu}\right) &= \int_{\tau(E)} x^*\phi d\overline{\mu} \\ &= \int_{\tau(E)} \overline{x^*f} d\overline{\mu} \\ &= \int_E x^*f d\mu \\ &= x^*\left((P)-\int_E f d\mu\right) \end{aligned}$$

and so the statement is proved.

Remark 5.3 From the previous result we have:

$$\begin{aligned}
(A)-\int_E F d\mu &= \left\{(P)-\int_E f d\mu, f \in S^1_{F,P}\right\} \\
&= \left\{(B)-\int_{\tau(E)} \overline{f} d\overline{\mu}, f \in S^1_{F,P}\right\} \\
&\subseteq (A)-\int_{\tau(E)} \overline{F} d\overline{\mu}
\end{aligned} \tag{2}$$

The aim of this section is to prove that (2) is in fact an equality. From now on we suppose that $L^1_\mu(X)$ is complete. There are several papers concerning the completeness of $L^1_\mu(X)$ in the finitely additive case; we mention for instance [4, 5, 16]. We give more details and examples in the last section.

Theorem 5.4 *Suppose that $F : \Omega \to cb(X)$ is a totally measurable and integrably bounded multifunction and $\psi \in S^1_{\overline{F}}$. Then there exists a Σ-measurable, Bochner integrable function f, such that its Stone extension ϕ is equal to ψ $\overline{\mu}$-almost everywhere and $\mu(\{\omega \in \Omega : d(f, F) \geq \alpha\}) = 0$ for all $\alpha > 0$.*

Proof: The proof is an adaption to the weakly sequentially compact case of that given in [20, Theorem 5.6].

It is known that if $C \in cb(X)$ then $cl\{C + \varepsilon X_1\} = C + \varepsilon X_1$ and $C = \bigcap_{\varepsilon>0}(C + \varepsilon X_1)$.

Theorem 5.5 *Let $F : \Omega \to cb(X)$ be totally measurable and integrably bounded multifunction. Then for all $E \in \Sigma$*

$$cl\left\{(A)-\int_E F d\mu\right\} = (A)-\int_{\tau(E)} \overline{F} d\overline{\mu},$$

where $\tau : \Sigma \to \mathcal{G}$, as in section 4, is the Stone isomorphism.

Proof: The right hand side is closed. The first inclusion follows from Remark 5.3. We now prove that

$$(A)-\int_{\tau(E)} \overline{F} d\overline{\mu} \subseteq cl\left\{(A)-\int_E F d\mu\right\}.$$

Let $\psi \in S^1_{\overline{F}}$, and $\alpha > 0$ be fixed. By Theorem 5.4 there exists $f \in L^1_\mu(X)$ such that the Stone extension ϕ of f is equal to ψ, $\overline{\mu}$-almost everywhere, and $f \in S^1_{F+\alpha X_1}$. Let Γ_f be as in Proposition 3.8; Γ_f is weakly measurable, integrably bounded and so, by Proposition 3.1, it admits Pettis integrable selections.
If $g \in S^1_{\Gamma_f,P}$, then $g \in S^1_{F,P}$ and μ-almost everywhere $\|f(\omega) - g(\omega)\| = r(\omega) \leq \alpha$. Moreover

$$\begin{aligned}
\|(P)-\int_E g d\mu - (P)-\int_E f d\mu\| &= \sup_{x^* \in X_1^*} \left| x^*\left((P)-\int_E g d\mu\right) - x^*\left((P)-\int_E f d\mu\right)\right| \leq \\
&\leq \sup_{x^* \in X_1^*} \int_E |x^*(g-f)| d\mu \leq \alpha\mu(E)X_1.
\end{aligned}$$

Then

$$
\begin{aligned}
(B)-\int_{\tau(E)} \psi d\overline{\mu} &= (B)-\int_{\tau(E)} \phi d\overline{\mu} \\
&= (P)-\int_E f d\mu \in (A)-\int_E (F+\alpha X_1) d\mu \\
&\subseteq cl\left\{(A)-\int_E F d\mu + \alpha\mu(E) X_1\right\}.
\end{aligned}
$$

In fact, $f(\omega) = f(\omega) - g(\omega) + g(\omega)$ and $f - g \in S^1_{\alpha X_1}$; therefore

$$
\begin{aligned}
(P)-\int_E f d\mu = (P)-\int_E (f-g) d\mu + (P)-\int_E g d\mu & \\
&\in \left\{\alpha\mu(E)X_1 + (A)-\int_E F d\mu\right\}.
\end{aligned}
$$

Since $cl\left\{(A)-\int_E F dm\right\}$ belongs to $cb(X)$, we obtain

$$
\int_{\tau(E)} \psi d\overline{\mu} \in cl\left\{(A)-\int_E F d\mu\right\} + \alpha\mu(E)X_1
$$

and so, by the arbitrariness of $\alpha > 0$,

$$
\int_{\tau(E)} \psi d\overline{\mu} \in cl\left\{(A)-\int_E F d\mu\right\}
$$

and the statement is proved.

We are now ready to prove that the Aumann integral of F is closed. Let $j : S^1_{F,P} \to L^1_{\overline{\mu}}(X)$ be the map which sends every Pettis integrable selection into its Stone extension.

Proposition 5.6 *The set $j(S^1_{F,P})$ is convex and closed in $L^1_{\overline{\mu}}(X)$.*

Proof: The convexity follows from the fact that F is convex valued, while the closedness may be proved in the same way as in [20, Lemma 5.1].

We recall that the dual of $L^1_{\overline{\mu}}(X)$ is $L^\infty_{\overline{\mu}}(X^*)$ (see for example [12]), and we denote by $\sigma(L^1, L^\infty)$ the weak topology on $L^1_{\overline{\mu}}(X)$.

It is well known that

Proposition 5.7 *The operator $T : L^1_{\overline{\mu}}(X) \to X$ defined by*

$$
T(\phi) = \int_S \phi d\overline{\mu},
$$

is continuous with respect to the $(\sigma(L^1, L^\infty), \sigma(X, X^\star))$ topology.

Proposition 5.8 *$S^1_{\overline{F}}$ is weakly compact in $L^1_{\overline{\mu}}(X)$.*

Proof: This follows now from [17, Theorem 3.7].

Proposition 5.9 *The integral* $(A) - \int F d\mu$ *is closed.*

Proof: $j(S^1_{F,P})$ is closed and convex, therefore weakly closed in $S^1_{\overline{F}}$ which in turn is weakly compact. Thus $j(S^1_{F,P})$ is weakly compact , as well, and so $T(j(S^1_{F,P})) = (A) - \int F d\mu$ is closed in X.

Combining Theorem 5.5 and Proposition 5.9 we arrive at our main result.

Theorem 5.10 *Suppose* (Ω, Σ) *is a measurable space,* μ *is a bounded finitely additive measure,* Σ *is a* μ*-complete* σ*-algebra and* $L^1_\mu(X)$ *is complete. If* $F \in L^1(\Omega, \Sigma, \mu, cb(X))$*, then*

$$(A) - \int_E F d\mu = (D) - \int_E F d\mu.$$

Proof: We recall that if $F \in L^1(\Omega, \Sigma, \mu, cb(X))$, then

$$(A) - \int_{\tau(E)} \overline{F} d\overline{\mu} = (D) - \int_{\tau(E)} \overline{F} d\overline{\mu},$$

(see [6]). Moreover $\overline{F} \in L^1(S, \mathcal{G}_\sigma, \overline{\mu}, cb(X))$ and

$$(D) - \int_E F d\mu = (D) - \int_{\tau(E)} \overline{F} d\overline{\mu}.$$

So, by Theorem 5.5 and Proposition 5.9, the Debreu and Aumann integrals of the multifunction F coincide.

6 Examples

We will exhibit a series of examples of spaces (Ω, Σ, μ), where μ is only finitely additive, and Σ is a μ-complete σ-algebra, with $L^1_\mu(\mathbb{R})$ complete.

We begin with some definitions.
Two nonnegative finitely additive measures μ, λ are *strongly singular* if there exists a set $Q \in \Sigma$ such that $\lambda(Q) = \mu(\Omega \setminus Q) = 0$. When Σ is a σ-algebra the strong singularity of a pair of nonnegative finitely additive measures is equivalent to their separability, (see for example [4, Proposition 8.4]). A nonnegative finitely additive measure μ is *continuous* if for every $\varepsilon > 0$ there exists a finite decomposition of Ω, say $(A_1, \cdots, A_n) \in \Sigma$, such that $\mu(A_i) \leq \varepsilon$ for every $i = 1, \cdots, n$.

Example 6.1 An example of a finitely additive space (Ω, Σ, μ) with Σ μ-complete and $L^1_\mu(X)$ complete can be found in [5, Example 2.1.3.4 and Remark 4.6.8] . In this example Ω is countable and μ is superatomic in the sense of [5, Definition 5.3.4].

Example 6.2 This example is due to D. Candeloro. Let $\mathcal{U}$ be an ultrafilter on $[0, 1]$ and μ the finitely additive ultrafilter measure defined in $\mathcal{P}([0, 1])$ (i.e., μ takes value 1 on every set of the ultrafilter $\mathcal{U}$

and 0 otherwise). Is is easy to see that a function $f : [0,1] \to I\!R$ is μ-integrable if and only if f is μ-essentially bounded and that $\|f\|_1 = \|f\|_\infty$; in fact,

$$\begin{aligned}\int_0^1 |f|d\mu &= \int_0^\infty \mu(\{x : |f(x)| > t\})dt \\ &= \sup\{t \geq 0 : \{|f| > t\} \in \mathcal{U}\} \\ &= \inf\{t > 0 : \mu(\{|f| > t\}) = 0\}.\end{aligned}$$

So $L^1_\mu(I\!R) = L^\infty_\mu(I\!R)$ and it is well known that $L^\infty_\mu(X)$ is complete (see, for example, [7]).
On the other hand, we can obtain the same result observing that μ has Radon-Nikodym Property, i.e. for every finitely additive measure ν which is absolutely continuous with respect to μ in the $\varepsilon - \delta$ sense there exists $f \in L^1_\mu(I\!R)$ such that $\nu(\cdot) = \int_\cdot f d\mu$. In fact, if $\nu \ll \mu$ then $\nu(E) = 0$ if $\mu(E) = 0$ and then $\nu([0,1]) = \nu(B)$ for every $B \in \mathcal{U}$. Thus, if $u = \nu([0,1])$, $\nu = u\mu$ and $d\nu/d\mu = u$. Following [4, Theorem 7.5] we obtain the result. Here again μ is superatomic but Ω is not countable.

Example 6.3 Let $\Omega = [0,1]$, Q the set of rational numbers in $[0,1]$, and $\mathcal{U}$ an ultrafilter of Borel sets containing Q and which does not contain the singletons. Let μ be the corresponding ultrafilter finitely additive measure in $\mathcal{B}([0,1])$. As in the previous example, $L^1_\mu(I\!R)$ is complete. We consider now $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is the Lebesgue measure. Since μ and λ are nonnegative and strongly singular, $L^1_{\mu+\lambda}(I\!R)$ is complete by [4, Propositions 8.4 and 8.6]. In this case μ is atomic but not superatomic.

Example 6.4 If Σ is a σ-algebra and $L^1_\mu(X)$ is complete, then $\widetilde{\Sigma}$, as defined in [4, Section 1.6], coincides with the completion of Σ, and $L^1_\mu(X) = L^1_{\widetilde{\mu}}(X)$, where $\widetilde{\mu}$ is the Peano-Jordan extension of μ to $\widetilde{\Sigma}$.

Finally we give an example of a class of measures spaces which satisfy the previous hypotheses and for which μ is also continuous. It was pointed out to us by P. Berti and P. Rigo and can be found in [15].

Example 6.5 Let F be the set of non null integers, $F = I\!N \cup (-I\!N)$ and H the family of sequences in F. We consider the discrete topology in F and the product topology in H. Let $\mathcal{B}(H)$ be the Borel σ-algebra on H. Let $\mu_1, \mu_2 : \mathcal{P}(F) \to \{0,1\}$ be two finitely additive measures such that $\mu_i(\{n\}) = 0$ for every n and for $i = 1,2$ and $\mu_1(I\!N) = \mu_2(-I\!N) = 1$. Put $\mu = \dfrac{\mu_1 + \mu_2}{2}$. From μ it is possible to construct $\nu : \mathcal{B}(H) \to [0,1]$ with suitable properties and such that for every $n \in I\!N$ and for every $C_1, C_2, \cdots, C_n \subset F$ one has

$$\nu(C_1 \times C_2 \times \cdots \times C_n \times F \times F \times \cdots) = \mu(C_1)\cdots\mu(C_n).$$

Then ν is continuous and $L^1_\nu(I\!R)$ is complete.

Example 6.6 This example is due to H. Weber. Let $\lambda : \mathcal{M} \to [0,1]$ be the Lebegue measure on the σ-algebra of all Lebesgue measurable subsets of $[0,1]$ and $\mu : \mathcal{P}([0,1]) \to [0,1]$ a finitely additive extension of λ on the power set of $[0,1]$ such that $\mathcal{A}$ is dense in the semimetric space $(\mathcal{P}([0,1]), d_\mu)$; here d_μ is defined by $d_\mu(A,b) = \mu(A\Delta B)$. Such an extension exists by a theorem of Z. Lipecki [19]. The measure μ is not σ-additive since λ has no σ-additive extension on $\mathcal{P}([0,1])$, but μ is continuous since its restriction λ is continuous. $(\mathcal{P}([0,1]), d_\mu)$ is complete since $(\mathcal{M}, d_\lambda)$ is a complete dense subspace of $(\mathcal{P}([0,1]), d_\mu)$. Using the following theorem one obtains that $L^1_\mu(I\!R)$ is complete.

Theorem ([10]). Let $\nu : \mathcal{A} \to [0,+\infty)$ be a finitely additive measure on an algebra of subsets of Ω, $\nu^* : \mathcal{P}(\Omega) \to [0,+\infty[$ the outer measure of ν, and $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$ in $(\mathcal{P}(\Omega), d_{\nu^*})$. Then $L^1_\nu(I\!R)$ is complete if and only if $(\overline{\mathcal{A}}, d_{\nu^*})$ is complete.

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Received 07.11.2001