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Some properties of an improper GHk integral in Riesz spaces

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Abstract

We investigate the GH_k integral for functions defined on (possibly) unbounded subintervals of the extended real line and with values in Riesz spaces. Some convergence theorems are proved, together with a version of the Fundamental Formula of Calculus.

1 Introduction.

In the literature there are several studies concerning Stieltjes-type integrals and their generalizations. In [23, 24, 25], and with a more natural and transparent approach in [14, 15], an abstract integral (GH_k integral) for real-valued functions defined in a compact subinterval of the real line has been studied, extending the "generalized Perron integral" investigated by Š. Schwabik in [33], which precisely

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corresponds to the GH_1 integral. This last integral has as particular cases also the classical Kurzweil-Henstock and Henstock-Stieltjes integrals: indeed it is enough to take functions of the type

$$U(t,x) = f(t) \cdot g(x), \qquad t, x \in [a,b], \text{ with } a, b \in \mathbb{R}.$$

The Kurzweil-Henstock integral for Riesz space-valued functions was introduced and investigated in [1, 26, 27, 28, 29, 30, 31]. In particular, the Kurzweil-Henstock integral for functions defined in unbounded intervals of the extended real line and with values in Riesz spaces, Banach spaces, metric semigroups was studied in [5, 8, 9] respectively. The Kurzweil-Henstock integral for maps defined in abstract topological spaces was investigated in [3] for real-valued functions and in [4, 6] for Riesz space-valued functions.

In the GH_k integral, instead of functions of two variables, corresponding maps U of $k+1$ variables are taken, and in [14, 15] some examples of other concrete cases of possible choices of U are illustrated: for instance the fundamental tool of the divided differences, some various versions of the k-th derivative, k-variation and k-convexity, and a short history and bibliography about the Stieltjes-type integrals studied in the literature, for which the considered GH_k integral is an extension. These tools are furthermore useful in the literature also in order to study the Perron integral of order k and its fundamental properties (see for example [2, 10, 11, 13, 19]).

In this paper we generalize to the case of Riesz space-valued functions, defined on (possibly) *unbounded* subintervals of the extended real line, the GH_k integral investigated and we then extend the main properties, the Hake's theorem, the Saks-Henstock Lemma and the Fundamental Formula of Calculus proved in [14, 15]. Furthermore we give some versions of the monotone and dominated convergence theorems.

2 Preliminaries.

Definition 2.1 A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

Definition 2.2 A bounded double sequence $(a_{i,j})_{i,j}$ in R is called *regulator* or (D)-*sequence* if, for each $i \in \mathbb{N}$, $a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1}\ \forall\, j \in \mathbb{N}$ and $\bigwedge_{j\in\mathbb{N}} a_{i,j} = 0$.

Given a sequence $(r_n)_n$ in R, we say that $(r_n)_n$ (D)-*converges* to an element $r \in R$ if there exists a regulator $(a_{i,j})_{i,j}$, satisfying the following condition:

$$\text{for all maps } \varphi \in \mathbb{N}^{\mathbb{N}}, \text{there exists an integer } n_0 \text{ such that}$$

$$|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \geq n_0$. In this case, we write $(D)\lim_n r_n = r$.

Analogously, given $l \in R$, a function $f : A \to R$, where $\emptyset \neq A \subset \widetilde{\mathbb{R}}$, and a limit point x_0 for A, we will say that $(D)\lim_{x\to x_0} f(x) = l$ if there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a neighborhood $\mathcal{U}$ of x_0 such that for all $x \in \mathcal{U} \cap A \setminus \{x_0\}$ we get

$$|f(x) - l| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Definition 2.3 We say that R is *weakly σ-distributive* if for every (D)-sequence $(a_{i,j})_{i,j}$ one has:

$$\bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0. \tag{1}$$

Throughout the paper, we shall always assume that R is a Dedekind complete weakly σ-distributive Riesz space.

The following lemma will be useful in the sequel.

Lemma 2.4 ([27], pp. 42-43) *Let* $\{(a_{i,j}^{(p)})_{i,j} : p \in \mathbb{N}\}$ *be any countable family of regulators. Then for each fixed element* $u \in R$, $u \geq 0$, *there exists a regulator* $(a_{i,j})_{i,j}$ *such that, for every* $\varphi \in \mathbb{N}^{\mathbb{N}}$,

$$u \wedge \sum_{p=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+p)}^{(p)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

We now extend some basic concepts of [14] to the case of functions defined on unbounded intervals of the extended real line. From now on we suppose that $a, b \in \widetilde{\mathbb{R}}$, with $a < b$, unless we state differently. As usual, we set $[a, b] := \{x \in \widetilde{\mathbb{R}} : a \leq x \leq b\}$, $]a, b[:= \{x \in \widetilde{\mathbb{R}} : a < x < b\}$ and we denote by (a, b) an interval which may or not contain its endpoints.

Definitions 2.5 Let $k \in \mathbb{N}$ be fixed. Set

$$a \leq x_{1,0} < \ldots < x_{1,k} \leq x_{2,0} < \ldots < x_{2,k} \leq \ldots \leq x_{n,0} < x_{n,1} < \ldots < x_{n,k} \leq b,$$

and $\xi_i \in [x_{i,0}, x_{i,k}]$, $i = 1, \ldots, n$. We say that the intervals $[x_{i,0}, x_{i,k}]$ form a *tagged k-decomposition,* or *k-decomposition* of (a, b), and denote it by the notation

$$\Pi := \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\}.$$

A k-decomposition of (a, b) is called *tagged k-partition* (or *k-partition*) if

$$\bigcup_{i=1}^{n} [x_{i,0}, x_{i,k}] = (a, b).$$

A *gauge* is a map γ defined in (a, b) and taking values in the set of all open intervals in $\widetilde{\mathbb{R}}$, such that $\xi \in \gamma(\xi)$ for every $\xi \in (a, b)$; moreover we require $\gamma(\xi)$ to be bounded as soon as $\xi \in \mathbb{R} \cap (a, b)$. Given a gauge γ, a k-decomposition of (a, b) of the type

$$\Pi = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\} \tag{2}$$

is said to be γ-*fine* if $\xi_i \in [x_{i,0}, x_{i,k}] \subset \gamma(\xi_i)$ for all $i = 1, \ldots, n$. Observe that for any gauge γ there always exists a γ-fine k-partition (see also [14, 21]).

Definition 2.6 Given a bounded interval $[a,b] \subset \mathbb{R}$ and a map $\delta : [a,b] \to \mathbb{R}^+$, a partition Π of $[a,b]$ as in (2) is said to be δ-*fine* if $\xi_i \in [x_{i,0}, x_{i,k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \dots, n$.

Remark 2.7 Observe that, if $[x_{i,0}, x_{i,k}]$ is an unbounded interval of a γ-fine partition, then the element ξ_i associated with $[x_{i,0}, x_{i,k}]$ is necessarily $+\infty$ or $-\infty$: otherwise $\gamma(\xi_i)$ should be a bounded interval and contain an unbounded interval, a contradiction.

Definition 2.8 Given any k-decomposition of (a,b),

$$\Pi = \{(\xi_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i,0}, \dots, x_{i,k}], i = 1, \dots, n\}$$

and a function $U : (a,b)^{k+1} \to R$, we call *Riemann sum* of U (and we write $\sum_{\Pi} U$) the expression

$$\sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1})]. \tag{3}$$

We now formulate our definition of GH_k integral for R-valued functions defined on $(a,b)^{k+1}$. We will show that our definition can be formulated equivalently both with gauges and with positive maps δ.

Definition 2.9 We say that a function $U : (a,b)^{k+1} \to R$ is GH_k *integrable* on (a,b) if there exist $I \in R$ and a (D)-sequence $(a_{i,j})_{i,j}$ in R such that to all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there correspond a function $\delta : [c,d] \to \mathbb{R}^+$ and a positive real number P such that

$$\left| \sum_{\Pi} U - I \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{4}$$

whenever Π is a δ-fine k-partition of any bounded interval $[c,d]$ with $[c,d] \supset (a,b) \cap [-P,P]$. In this case we say that I is the GH_k *integral of* U, and we denote the element I by the symbol $(GH_k) \int_a^b U$, writing usually $U \in GH_k(a,b)$.

Analogously it is possible to define the integral for every subinterval of (a,b).

Remark 2.10 **a)** We note that the GH_k integral is well-defined, that is there exists at most one element I, satisfying condition (4) (see also [5], Remark 3.4).

b) It is readily seen that, when both a and b belong to $\mathbb{R}$, the definition of the GH_k integral is equivalent to the (more "classical") one in which only maps of the type $\delta : (a,b) \to \mathbb{R}^+$ are involved (see also [21]).

c) If $[a,b] \subset \widetilde{\mathbb{R}}$, $R = \mathbb{R}$, $k = 1$, $f : [a,b] \to \mathbb{R}$ and $U(t,x) = f(t) \cdot x$ for $t, x \in [a,b]$, $x \neq \pm\infty$, then we obtain the classical improper integral (see also [33], p. 4).

We now give the following characterization of the GH_k integrability.

Theorem 2.11 *A function $U : (a,b)^{k+1} \to R$ is GH_k integrable if and only if there exist $J \in R$ and a (D)-sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge γ such that*

$$\left| \sum_{\Pi} U - J \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{5}$$

whenever Π is a γ-fine partition of (a,b), and in this case we have $\int_a^b U = J$.

Proof: See also [5], Theorem 3.3.

3 Elementary properties of the improper GH_k integral

The proof of the following proposition is straightforward (see also [5]).

Proposition 3.1 *If $U_1, U_2 \in GH_k(a,b)$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 U_1 + c_2 U_2 \in GH_k(a,b)$, and*

$$(GH_k) \int_a^b (c_1 U_1 + c_2 U_2) = c_1 (GH_k) \int_a^b U_1 + c_2 (GH_k) \int_a^b U_2;$$

if $U, V \in GH_k(a,b)$ and $U \leq V$, then

$$(GH_k) \int_a^b U \leq (GH_k) \int_a^b V;$$

if $U, |U| \in GH_k(a,b)$*, then*

$$\left|(GH_k)\int_a^b U\right| \leq (GH_k)\int_a^b |U|.$$

We now state the Cauchy criterion.

Theorem 3.2 *A map* $U : (a,b)^{k+1} \to R$ *is* GH_k *integrable if and only if there is a* (D)*-sequence* $(a_{i,j})_{i,j}$ *in* R *such that for all* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *there exists a gauge* $\gamma = \gamma(\varphi)$ *such that for all* γ*-fine* k*-partitions* Π, Π' *of* (a,b) *we have*

$$\left|\sum_{\Pi} U - \sum_{\Pi'} U\right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Proof: Straightforward. □

We now investigate GH_k integrability on subintervals.

Proposition 3.3 *If* $U \in GH_k(a,b)$*, then* $U \in GH_k(c,d)$ *for each* $(c,d) \subset (a,b)$ *with respect to a same regulator, independent on* (c,d)*.*

Proof: Without loss of generality, we suppose that $(c,d) = (a,d)$, with $a < d < b$.

Let γ be any gauge on (a,b), pick any two γ-fine k-partitions Π_1, Π_2 of (a,d), and let Π' be a γ-fine k-partition of (d,b). Such a partition does exist, by virtue of the Cousin lemma. Then, for $j = 1,2$, $\Pi''_j := \Pi' \cup \Pi_j$ is a γ-fine partition of (a,b). Since

$$\left|\sum_{\Pi_1} U - \sum_{\Pi_2} U\right| = \left|\sum_{\Pi''_1} U - \sum_{\Pi''_2} U\right|,$$

then the assertion follows from the Cauchy criterion. □

Remark 3.4 Note that this proof shows that, if a regulator works for GH_k integrability on (a,b), then it works for integrability of (a,c) for every $a < c < b$; this will be useful in the sequel.

Corollary 3.5 *If* $U \in GH_k(a,b)$ *and* $a < c < b$*, then*

$$(GH_k)\int_a^b U = (GH_k)\int_a^c U + (GH_k)\int_c^b U.$$

Proof: Straightforward. □

We now introduce the following:

Definition 3.6 Let $U : (a,b)^{k+1} \to R$ and fix a point $x_0 \in (a,b)$. We say that U is *continuous at* x_0 *uniformly with respect to* $t_1, \ldots, t_k$ if there is a (D)-sequence $(d_{i,j})_{i,j}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\eta(x_0) \in \mathbb{R}^+$ such that

$$|U(x_0; t_1, \ldots, t_k) - U(x_0; t'_1, \ldots, t'_k)| \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}$$

whenever $t_j, t'_l \in (a,b)$ with $0 < |t_j - x_0| \leq \eta(x_0)$, $0 < |t'_l - x_0| \leq \eta(x_0)$, $j, l = 1, \ldots, k$ (see also [16], Section 3, pp. 138-139).

Let $x_0 \in]a,b[$. We say that U satisfies condition $[H1)$ at $x_0]$ if there exists a (D)-sequence $(c_{i,j})_{i,j}$ (depending in general on the chosen point x_0) such that to all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive real number $\eta = \eta(x_0)$ such that

$$\begin{aligned} &\Big| [U(x_0; w_1^{(0)}, \ldots, w_k^{(0)}) - U(x_0; w_0^{(0)}, \ldots, w_{k-1}^{(0)})] \\ - \quad & [U(x_0; w_1^{(1)}, \ldots, w_k^{(1)}) - U(x_0; w_0^{(1)}, \ldots, w_{k-1}^{(1)})] \\ - \quad & [U(x_0; w_1^{(2)}, \ldots, w_k^{(2)}) - U(x_0; w_0^{(2)}, \ldots, w_{k-1}^{(2)})] \Big| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \end{aligned}$$

whenever $\bigcup_{l=0}^{2} \left(\bigcup_{i=1}^{k} [w_{i-1}^{(l)}, w_i^{(l)}] \right) \subset]x_0 - \eta, x_0 + \eta[$ and $w_0^{(0)} = w_0^{(1)}$, $w_k^{(0)} = w_k^{(2)}$, $x_0 = w_k^{(1)} = w_0^{(2)}$.

Remark 3.7 Observe that for $k = 1$ condition H1) is automatically satisfied, because each term of the involved Riemann sums is formed by the difference of two values of the function U (see also [33], Theorem 1.11, pp. 10-12).

Moreover, note that, when $R = \mathbb{R}$, property H1) is implied by the condition of "existence of the iterated limit $J(U, c)$" used by A. G. Das and S. Kundu (see [14], Definition 2.9., p. 69) when $k \geq 2$. Finally, observe that property $H1)$ at x_0 holds whenever U is continuous at x_0 uniformly with respect to $t_1, \ldots, t_k$.

We are now ready to prove the following result on additivity.

Theorem 3.8 *Let $k \geq 2$, and $U : (a,b)^{k+1} \to R$ be a function which satisfies condition $H1)$ at the point $c \in]a,b[$. If $U \in GH_k(a,c)$ and $U \in GH_k(c,b)$, then $U \in GH_k(a,b)$ and*

$$(GH_k)\int_a^b U = (GH_k)\int_a^c U + (GH_k)\int_c^b U.$$

Proof: By the hypotheses it follows that there is a (D)-sequence $(e_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a positive function δ^* and a real number P (without loss of generality, $P > |c|$) with the following property: for each δ^*-fine k-partition Π_1 of any bounded interval $[a_1, b_1] \subset (a,c)$, $[a_1,b_1] \supset (a,c) \cap [-P,P]$ and Π_2 of every bounded interval $[a_2,b_2] \subset (c,b)$, $[a_2,b_2] \supset (c,b) \cap [-P,P]$ we get

$$\left|\sum_{\Pi_1} U - (GH_k)\int_a^c U\right| \leq \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}, \quad \left|\sum_{\Pi_2} U - (GH_k)\int_c^b U\right| \leq \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}.$$

Let $(c_{i,j})_{i,j}$ and $\eta(c) = \eta(c)(\varphi)$ be related with condition $H1)$ at c, and define a function δ on (a,b) by setting $\delta(x) = \min\{\delta^*(x), |x-c|\}$ if $x \in (a,b) \setminus \{c\}$, and $\delta(c) = \min\{\delta^*(c), \eta(c)\}$. Pick now any bounded interval $[\alpha,\beta] \subset (a,b)$, $[\alpha,\beta] \supset (a,b) \cap [-P,P]$, and any δ-fine k-partition

$$\Pi = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\}$$

of $[\alpha,\beta]$. There exists m, with $1 \leq m \leq n$, such that $c = \xi_m$, and no other interval but $[x_{m,0}, x_{m,k}]$ can contain c. We get:

$$\begin{aligned}\sum_{\Pi} U &= \sum_{i=1}^{m-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] \\ &+ [U(c; x_{m,1}, \ldots, x_{m,k}) - U(c; x_{m,0}, \ldots, x_{m,k-1})] \\ &+ \sum_{i=m+1}^{n} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})].\end{aligned}$$

Consider now the points

$$c - \delta(c) < x_{m-1,k} = y_{m,0} < \ldots < y_{m,k} = c = z_{m,0} < \ldots < z_{m,k} = x_{m+1,0} < c + \delta(c).$$

The parts of the partition Π for $i = 1, \dots, m-1$ ($i = m+1, \dots, n$) and the single system $\{(c; y_{m,1}, \dots, y_{m,k-1}) : [y_{m,0}, c]\}$ ($\{(c; z_{m,1}, \dots, z_{m,k-1}) : [c, z_{m,k}]\}$) form a δ^*-fine k-partition Π_1 (Π_2) of $[\alpha, c]$ ($[c, \beta]$). We have:

$$\begin{aligned}\left|\sum_{\Pi} U - \sum_{\Pi_1} U - \sum_{\Pi_2} U\right| &= |[U(c; x_{m,1}, \dots, x_{m,k}) - U(c; x_{m,0}, \dots, x_{m,k-1})] \\ &- [U(c; y_{m,1}, \dots, y_{m,k} = c) - U(c; y_{m,0}, \dots, y_{m,k-1})] \\ &- [U(c; z_{m,1}, \dots, z_{m,k}) - U(c; z_{m,0} = c, \dots, z_{m,k-1})]| \leq \\ &\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\left|\sum_{\Pi} U - (GH_k)\int_a^c U - (GH_k)\int_c^b U\right| &\leq \left|\sum_{\Pi_1} U - (GH_k)\int_a^c U\right| + \left|\sum_{\Pi_2} U - (GH_k)\int_c^b U\right| \\ &+ \left|\sum_{\Pi} U - \sum_{\Pi_1} U - \sum_{\Pi_2} U\right| \leq 2\bigvee_{i=1}^{\infty} e_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.\end{aligned}$$

From this it follows that $U \in GH_k(a, b)$ and

$$(GH_k)\int_a^b U = (GH_k)\int_a^c U + (GH_k)\int_c^b U. \qquad \square$$

4 Convergence theorems

We begin with a version of the Saks-Henstock lemma.

Lemma 4.1 *Let $U : (a, b)^{k+1} \to R$ be GH_k integrable on (a, b). Then there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge γ such that, whenever*

$$\Pi := \{(\eta_i; y_{i,1}, \dots, y_{i,k-1}) : [y_{i,0}, y_{i,k}], i = 1, \dots, m\} \tag{6}$$

is a γ-fine k-decomposition of (a, b) (where $y_{i-1,k} \leq y_{i,0}$ $(i = 2, \dots, m)$), then

$$\left|\sum_{i=1}^{m} \left[U(\eta_i; y_{i,1}, \dots, y_{i,k}) - U(\eta_i; y_{i,0}, \dots, y_{i,k-1}) - (GH_k)\int_{y_{i,0}}^{y_{i,k}} U\right]\right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Proof: Let $(a_{i,j})_{i,j}$ be a (D)-sequence, related with GH_k integrability of U on (a,b), pick arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$, and take a gauge γ in correspondence with φ, whose existence is guaranteed by Theorem 2.11. Let the $y_{i,l}$'s be as in (6). If $y_{i,k} < y_{i+1,0}$ for some $i = 1,\dots,m$, $y_{m+1,0} = b$, then, by Proposition 3.3, $U \in GH_k[y_{i,k}, y_{i+1,0}]$. Since the involved i's are a finite number, there exists a (D)-sequence $(b_{i,j})_{i,j}$ such that for every $\psi \in \mathbb{N}^{\mathbb{N}}$ and $i = 1,\dots,m$ there is a gauge γ_i on $[y_{i,k}, y_{i+1,0}]$ such that $\gamma_i(x) \subset \gamma(x)$ for all $i = 1,\dots,m$ and each $x \in [y_{i,k}, y_{i+1,0}]$, and with the property that

$$\sum_{i=1}^{m} \left| \sum_{\Pi_i} U - (GH_k) \int_{y_{i,k}}^{y_{i+1,0}} U \right| \le \bigvee_{r=1}^{\infty} b_{r,\psi(r)} \tag{7}$$

for every γ_i-fine k-partition Π_i of $[y_{i,k}, y_{i+1,0}]$. If $y_{i,k} = y_{i+1,0}$, we obviously take $\sum_{\Pi_i} U = 0$. The quantity

$$\sum_{i=1}^{m} [U(\eta_i; y_{i,1},\dots,y_{i,k}) - U(\eta_i; y_{i,0},\dots,y_{i,k-1})] + \sum_{i=1}^{m} \left(\sum_{\Pi_i} U \right)$$

is a Riemann sum which corresponds to a certain γ-fine k-partition of (a,b), and hence we get:

$$\left| \sum_{i=1}^{m} [U(\eta_i; y_{i,1},\dots,y_{i,k}) - U(\eta_i; y_{i,0},\dots,y_{i,k-1})] + \sum_{i=1}^{m} \left(\sum_{\Pi_i} U \right) - (GH_k) \int_a^b U \right|$$
$$\le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

From this, (7) and Corollary 3.5 it follows that

$$\left| \sum_{i=1}^{m} \left[U(\eta_i; y_{i,1},\dots,y_{i,k}) - U(\eta_i; y_{i,0},\dots,y_{i,k-1}) - (GH_k) \int_{y_{i,0}}^{y_{i,k}} U \right] \right|$$
$$\le \left| \sum_{i=1}^{m} [U(\eta_i; y_{i,1},\dots,y_{i,k}) - U(\eta_i; y_{i,0},\dots,y_{i,k-1})] + \sum_{i=1}^{m} \left(\sum_{\Pi_i} U \right) - (GH_k) \int_a^b U \right|$$
$$+ \sum_{i=1}^{m} \left| \sum_{\Pi_i} U - (GH_k) \int_{y_{i,k}}^{y_{i+1,0}} U \right| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{r=1}^{\infty} b_{r,\psi(r)}.$$

Since

$$\left|\sum_{i=1}^{m}\left[U(\eta_i;y_{i,1},\ldots,y_{i,k})-U(\eta_i;y_{i,0},\ldots,y_{i,k-1})-(GH_k)\int_{y_{i,0}}^{y_{i,k}}U\right]\right|-\bigvee_{i=1}^{\infty}a_{i,\varphi(i)}\le\bigvee_{r=1}^{\infty}b_{r,\psi(r)}$$

for every $\psi\in\mathbb{N}^{\mathbb{N}}$, by weak σ-distributivity of R we obtain:

$$\left|\sum_{i=1}^{m}\left[U(\eta_i;y_{i,1},\ldots,y_{i,k})-U(\eta_i;y_{i,0},\ldots,y_{i,k-1})-(GH_k)\int_{y_{i,0}}^{y_{i,k}}U\right]\right|-\bigvee_{i=1}^{\infty}a_{i,\varphi(i)}\le 0.$$

This concludes the proof. □

We now prove a version of a Hake's theorem, which is an extension of the Cauchy theorem.

Theorem 4.2 *Let $a\in\mathbb{R}^+$, $U:(a,b)^{k+1}\to R$ be such that $U\in GH_k(a,c)$ for every $c\in(a,b[$. Assume that:*

H2) there are an element $I\in R$ and a (D)-sequence $(\alpha_{i,j})_{i,j}$ such that to every $\varphi\in\mathbb{N}^{\mathbb{N}}$ there corresponds a left neighborhood $\mathcal{U}$ of b such that

$$\left|(GH_k)\int_a^c U-I+U(b;y_1,\ldots,y_{k-1},b)-U(b;y_0,\ldots,y_{k-1})\right|\le\bigvee_{i=1}^{\infty}\alpha_{i,\varphi(i)}$$

whenever $\mathcal{U}\ni c\le y_0<y_1<\ldots<y_{k-1}<b$.

Moreover, suppose that

H3) there exist $u\in R$, $u\ge 0$, and a gauge γ_0, such that for every c with $a<c<b$ and for each γ_0-fine k-partition Π of $[a,c]$ we have:

$$\left|\sum_{\Pi}U-(GH_k)\int_a^c U\right|\le u.$$

Then $U\in GH_k(a,b)$ and $(GH_k)\displaystyle\int_a^b U=I$.

Furthermore, if $U\in GH_k[a,b]$, then $(D)\displaystyle\lim_{c\to b^-}(GH_k)\int_a^c U=(GH_k)\int_a^b U$ (this last result holds even if we drop both $H2)$ and $H3)$).

Remark 4.3 In general, in the first part of the assertion, hypothesis $H3$) cannot be dropped, even in the classical version of the Cauchy extension theorem for the classical Kurzweil-Henstock integral in Riesz spaces (see for instance [1], Example 4.21, and [5]). However, there are many situations in which $H3$) is satisfied, for example when $R = \mathbb{R}$ and $R = L^0(X, \mathcal{B}, \mu)$ with μ σ-additive and σ-finite (see also [4, 7]).

Proof of Theorem 4.2: Let $(c_p)_p$ be a strictly increasing sequence in $[a, b)$ with $c_p \uparrow b$ and $c_0 = a$. Thus for every $p \in \mathbb{N}$ there exists a (D)-sequence $(a^{(p)}_{i,j})_{i,j}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge $\gamma_p : [a, c_p] \to \mathbb{R}^+$, such that

$$\left| \sum_{\Pi_p} U - (GH_k) \int_a^{c_p} U \right| \leq \bigvee_{i=1}^{\infty} a^{(p)}_{i,\varphi(i+p)} \tag{8}$$

whenever Π_p is any γ_p-fine k-partition of $[a, c_p]$.

For every $\xi \in [a, b[$ there exists exactly one $p = p(\xi) \in \mathbb{N}$ such that $\xi \in [c_{p(\xi)-1}, c_{p(\xi)})$. Given $\xi \in [a, b[$, choose $\widehat{\gamma}(\xi)$ such that $\widehat{\gamma}(\xi) \subset \gamma_{p(\xi)}(\xi)$ and $\widehat{\gamma}(\xi) \cap [a, b[\subset [a, c_{p(\xi)})$. Let $c \in [a, b[$ and

$$\widehat{\Pi} := \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\}$$

be a $\widehat{\gamma}$-fine k-partition of $[a, c]$. For every $i = 1, \ldots, n$ we get:

$$[x_{i,0}, x_{i,k}] \subset \widehat{\gamma}(\xi_i) \subset [a, c_{p(\xi_i)}).$$

Moreover, $[x_{i,0}, x_{i,k}] \subset \gamma_{p(\xi_i)}(\xi_i)$. For every $p \in \mathbb{N}$, let us denote by the symbol

$$\sum_{i=1,\ldots,n, p(\xi_i)=p} \left[U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U \right]$$

the sum of those terms of

$$\sum_{i=1}^{n} \left[U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U \right]$$

for which $\xi_i \in [c_{p-1}, c_p)$. By Lemma 4.1 we obtain

$$\left| \sum_{i=1,\dots,n,\, p(\xi_i)=p} \left[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U \right] \right| \leq \bigvee_{i=1}^{\infty} a^{(p)}_{i,\varphi(i+p)}$$

for all $p \in \mathbb{N}$. Since $U \in GH_k[a, c]$ for every $c \in (a, b[$, then by Corollary 3.5 we have

$$(GH_k) \int_a^c U = \sum_{i=1}^{n} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U.$$

So we get:

$$\begin{aligned} & \left| \sum_{i=1}^{n} \left[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) \right] - (GH_k) \int_a^c U \right| \\ = & \left| \sum_{i=1}^{n} \left[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U \right] \right| \\ \leq & \sum_{p=1}^{\infty} \left| \sum_{i=1,\dots,n,\, p(\xi_i)=p} \left[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U \right] \right| \\ \leq & \sum_{p=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a^{(p)}_{i,\varphi(i+p)} \right). \end{aligned}$$

Furthermore, we have:

$$\left| \sum_{i=1}^{n} \left[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) \right] - (GH_k) \int_a^c U \right| \leq u,$$

where u is as in $H3$), since the involved k-partition $\widehat{\Pi}$ is γ_0-fine.

Let now $(b_{i,j})_{i,j}$ be a (D)-sequence such that

$$u \bigwedge \left(\sum_{p=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a^{(p)}_{i,\varphi(i+p)} \right) \right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \text{for all } \varphi \in \mathbb{N}^{\mathbb{N}} : \tag{9}$$

such a sequence does exist, by virtue of Lemma 2.4.

Let $(\alpha_{i,j})_{i,j}$ and $\mathcal{U}$ be related with condition $H2$), and pick a gauge γ on $[a, b]$ such that $\gamma(\xi) \subset \widehat{\gamma}(\xi)$ if $\xi \in [a, b[$, and $\gamma(b) \subset \mathcal{U}$. Let

$$\Pi := \{(\xi_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \dots, n\}$$

be any arbitrary γ-fine k-partition of $[a,b]$: we get $x_{n,k}=b$ and hence $\xi_n=b$ (otherwise we should get $[x_{n,0},x_{n,k}]\subset\widehat{\gamma}(\xi_n)\subset[a,c_{p(\xi_n)}]$ and thus $x_{n,k}<b$, a contradiction). Now we have:

$$\begin{aligned}\left|\sum_{\Pi}U-I\right| &\le \left|\sum_{i=1}^{n-1}[U(\xi_i;x_{i,1},\dots,x_{i,k})-U(\xi_i;x_{i,0},\dots,x_{i,k-1})]\right.\\ &+ \left.[U(b;x_{n,1},\dots,b)-U(b;x_{n,0},\dots,x_{n,k-1})]-I\right|\\ &\le \left|\sum_{i=1}^{n-1}[U(\xi_i;x_{i,1},\dots,x_{i,k})-U(\xi_i;x_{i,0},\dots,x_{i,k-1})]-(GH_k)\int_a^{x_{n-1,k}}U\right|\\ &+ \left|(GH_k)\int_a^{x_{n-1,k}}U-I+U(b;x_{n,1},\dots,b)-U(b;x_{n,0},\dots,x_{n,k-1})\right|\\ &\le \left|\sum_{i=1}^{n-1}[U(\xi_i;x_{i,1},\dots,x_{i,k})-U(\xi_i;x_{i,0},\dots,x_{i,k-1})]-(GH_k)\int_a^{x_{n-1,k}}U\right|\\ &+ \bigvee_{i=1}^{\infty}\alpha_{i,\varphi(i)}.\end{aligned}$$

As $x_{n-1,k}<b$ and $\widehat{\Pi}:=\{(\xi_i;x_{i,1},\dots,x_{i,k-1}):[x_{i,0},x_{i,k}],i=1,\dots,n-1\}$ is a $\widehat{\gamma}$-fine k-partition of $[a,x_{n-1,k}]$, we get

$$\left|\sum_{i=1}^{n-1}[U(\xi_i;x_{i,1},\dots,x_{i,k})-U(\xi_i;x_{i,0},\dots,x_{i,k-1})]-(GH_k)\int_a^{x_{n-1,k}}U\right|\le\bigvee_{i=1}^{\infty}b_{i,\varphi(i)},$$

and hence

$$\left|\sum_{\Pi}U-I\right|\le\bigvee_{i=1}^{\infty}b_{i,\varphi(i)}+\bigvee_{i=1}^{\infty}\alpha_{i,\varphi(i)}.$$

From this the first assertion follows.

We now turn to the last part. Since, by hypothesis, $U:[a,b]\to\mathbb{R}$ is GH_k integrable on $[a,b]$, then, thanks to Remark 3.4, U is GH_k integrable on $[a,c]$ for every $a<c\le b$ with respect to a *same* regulator $(a_{i,j})_{i,j}$, independent on the choice of the point c. Hence for all $\varphi\in\mathbb{N}^{\mathbb{N}}$ and $c\in(a,b]$ there exists $\delta_1^c:[a,c]\to\mathbb{R}^+$ such that for every δ_1^c-fine k-partition Π' of $[a,c]$ we get:

$$\left|\sum_{\Pi'}U-(GH_k)\int_a^c U\right|\le\bigvee_{i=1}^{\infty}a_{i,\varphi(i)}.$$

Moreover, thanks to the GH_k integrability on $[a,b]$, for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist $\delta : [a,b] \to \mathbb{R}^+$ and $P \in]a,b[$ such that for every bounded interval $[d_1,d_2] \subset [a,b]$ with $[d_1,d_2] \supset [-P,P]$ and for each δ-fine k-partition Π of $[d_1,d_2]$ we have

$$\left| \sum_{\Pi} U - (GH_k) \int_a^b U \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Let now $\varphi \in \mathbb{N}^{\mathbb{N}}$, $c > P$, $\delta_2^c(x) := \min\{\delta(x), \delta_1^c(x)\}$, $x \in [a,c)$, and Π be any δ_2^c-fine k-partition of $[a,c]$. Then we get:

$$\begin{aligned} \left|(GH_k) \int_a^c U - (GH_k) \int_a^b U\right| &\leq \left| \sum_{\Pi} U - (GH_k) \int_a^c U \right| + \left| \sum_{\Pi} U - (GH_k) \int_a^b U \right| \\ &\leq 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \end{aligned}$$

Thus the theorem is completely proved. □

Remark 4.4 An analogous version of Theorem 4.2 holds, if we consider, in our "limit operations" and calculus, the left endpoint instead of the right one.

Furthermore, in the literature several situations are investigated, when, in the Riemann sums, only the terms where the involved intervals are bounded are taken: this can be achieved by postulating it or by requiring that

$$U(\pm\infty; \lambda_1, \ldots, \lambda_k) = 0 \tag{10}$$

for every choice of $\lambda_j \in (a,b)$, $j = 1, \ldots, k$ (see also [5] and [21], p. 65).

Note that, when in the context $b = +\infty$ ($a = -\infty$) we assume (10), H2) can be replaced by the simpler condition of existence in R of the limit

$$(D) \lim_{c \to b^-} (GH_k) \int_a^c U. \tag{11}$$

Finally, observe that, when $R = \mathbb{R}$, H2) is implied by the two conditions of existence in $\mathbb{R}$ of the limit as in (11) and of "existence of the iterated limit (from the left) J^-" used by A. G. Das and S. Kundu (see [14]) when $k \geq 2$. For $R = \mathbb{R}$ and $k = 1$, H2) is equivalent to the condition formulated by Š. Schwabik ([33], formula (1.11), p. 15).

We will prove a version of the Beppo Levi monotone convergence theorem. We begin with a preliminary theorem.

Theorem 4.5 *Let $(U_n : (a,b)^{k+1} \to R)_n$ be a sequence of GH_k integrable functions. Suppose that:*

4.5.1) *there is a (D)-sequence $(b_{i,j})_{i,j}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a gauge ζ and $n_0 \in \mathbb{N}$ such that*

$$\left|(GH_k)\int_a^b U_n - \sum_{\Pi} U_n\right| \le \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for every ζ-fine k-partition Π and $n \ge n_0$;

4.5.2) *there exist: two functions $U_0 : (a,b)^{k+1} \to R$, $h^* : (a,b)^{k+1} \to \mathbb{R}^+$; a gauge γ_0^*; $w \in \mathbb{R}^+$; a (D)-sequence $(a^*_{i,j})_{i,j}$, such that:*
for every γ_0^*-fine k-partition

$$\Pi^* := \{(t_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \dots, q\}$$

we get

$$\sum_{i=1}^{q} h^*(t_i; x_{i,1}, \dots, x_{i,k}) \le w;$$

for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $t \in (a,b)$ there exists $p(t) \in \mathbb{N}$: $\forall\, n \ge p(t)$, whenever $\lambda_1, \dots, \lambda_k \in (a,b)$,

$$|U_0(t; \lambda_1, \dots, \lambda_k) - U_n(t; \lambda_1, \dots, \lambda_k)| \le h^*(t; \lambda_1, \dots, \lambda_k)\left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)}\right). \quad (12)$$

Then U_0 is GH_k integrable and

$$(D)\lim_n\,(GH_k)\int_a^b U_n = (GH_k)\int_a^b U_0.$$

Example 4.6 When $k = 1$, condition 4.5.2) is satisfied when $(U_n)_n$ converges to U_0 "with respect to the same regulator" and h^* is given by

$$h^*(t,\lambda) = \frac{\lambda}{1+t^2},\ t \in \mathbb{R}; \qquad h^*(\pm\infty, \lambda) = 0, \quad (13)$$

since the function $h(t) = \dfrac{1}{1+t^2}$, $t \in \mathbb{R}$, is Kurzweil-Henstock integrable on the whole of $\mathbb{R}$, and hence has bounded Riemann sums. We have introduced the function h^* substantially because we deal with unbounded intervals. Moreover, condition 4.5.2) is fulfilled, when $k = 1$, by h^* defined as in (13) and when $U_n(t,\lambda)$, $n \in \mathbb{N} \cup \{0\}$, is of the type $U_n(t,\lambda) = f_n(t) \cdot \lambda$, where the sequence of functions $(f_n)_n$ converges pointwise to f_0 "with respect to the same regulator" (see also [6]).

Proof of Theorem 4.5: We shall use the Cauchy criterion. Let $(b_{i,j})_{i,j}$, ζ and n_0 be as in 4.5.1). By 4.5.2) we get the existence of an element $w \in \mathbb{R}^+$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge $\eta \subset \zeta \cap \gamma_0^*$ (without loss of generality) such that, for every η-fine partition Π of (a,b), $\Pi = \{(t_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, q\}$, we have:

$$\begin{aligned}
\left| \sum_{\Pi} U_0 - \sum_{\Pi} U_n \right| &\leq \sum_{\Pi} |U_0(t_i; x_{i,1}, \ldots, x_{i,k}) - U_n(t_i; x_{i,1}, \ldots, x_{i,k}) \\
&- U_0(t_i; x_{i,0}, \ldots, x_{i,k-1}) + U_n(t_i; x_{i,0}, \ldots, x_{i,k-1})| \\
&\leq \sum_{i=1}^{q} h^*(t_i; x_{i,1}, \ldots, x_{i,k}) \left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} \right) + \qquad (14) \\
&+ \sum_{i=1}^{q} h^*(t_i; x_{i,0}, \ldots, x_{i,k-1}) \left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} \right) \\
&\leq 2\,w \left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} \right),
\end{aligned}$$

whenever $n \geq \max\{p(t_i) : i = 1, \ldots, q\}$. Put $a_{i,j} = 2\,w\,a^*_{i,j}$, $i,j \in \mathbb{N}$.

Without loss of generality, we can suppose that $p(t_i) \geq n_0 \; \forall\, i = 1, \ldots, n$. Choose now a (D)-sequence $(c_{i,j})_{i,j}$ such that

$$2 \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

Then for all η-fine k-partitions Π_1, Π_2, we have definitely:

$$\begin{aligned}\left|\sum_{\Pi_1} U_0 - \sum_{\Pi_2} U_0\right| \quad &\le \quad \left|\sum_{\Pi_1} U_0 - \sum_{\Pi_1} U_n\right| + \left|\sum_{\Pi_1} U_n - (GH_k)\int_a^b U_n\right| + \\ &+ \quad \left|(GH_k)\int_a^b U_n - \sum_{\Pi_2} U_n\right| + \left|\sum_{\Pi_2} U_n - \sum_{\Pi_2} U_0\right| \le \\ &\le \quad \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.\end{aligned}$$

GH_k integrability of U_0 follows from this and the Cauchy criterion.

By GH_k integrability of U_0 we obtain the existence of a (D)-sequence $(\overline{a}_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge η_1, depending on φ, such that

$$\left|(GH_k)\int_a^b U_0 - \sum_{\Pi} U_0\right| \le \bigvee_{i=1}^{\infty} \overline{a}_{i,\varphi(i)}$$

for every η_1-fine k-partition Π. By 4.5.1) there is a (D)-sequence $(b_{i,j})_{i,j}$ such that

$$\left|\sum_{\Pi} U_h - (GH_k)\int_a^b U_h\right| \le \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for every h greater than a suitable integer h_0 (depending on the involved φ) and for each η_2-fine k-partition Π. By 4.5.2), proceeding as in (14), we get the existence of a (D)-sequence $(c_{i,j})_{i,j}$ such that

$$\left|\sum_{\Pi} U_0 - \sum_{\Pi} U_h\right| \le \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for every $h \ge h'$, where h' is a positive integer depending on the involved k-partition Π. Without loss of generality, we can assume $h' \ge h_0$. Pick now a (D)-sequence $(d_{i,j})_{i,j}$ such that

$$\bigvee_{i=1}^{\infty} \overline{a}_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \le \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}.$$

Then (by arbitrariness of Π) we get:

$$\begin{aligned}\left|(GH_k)\int_a^b U_0 - (GH_k)\int_a^b U_h\right| &\le \left|(GH_k)\int_a^b U_0 - \sum_{\Pi} U_0\right| \\ + \quad \left|\sum_{\Pi} U_0 - \sum_{\Pi} U_h\right| &+ \left|\sum_{\Pi} U_h - (GH_k)\int_a^b U_h\right| \le \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}\end{aligned}$$

for every $h \geq h'$. We have proved that

$$(D)\lim_h (GH_k)\int_a^b U_h = (GH_k)\int_a^b U_0$$

and this concludes the proof. □

We now prove the monotone convergence theorem.

Theorem 4.7 *Let* $(U_n : (a,b)^{k+1} \to R)_n$ *be a sequence of* GH_k *integrable functions,* $U_n \leq U_{n+1}\,(n \in \mathbb{N})$*, and let the sequence* $\left((GH_k)\int_a^b U_n\right)_n$ *be bounded. Suppose that there exist two functions* U_0 *and* h^* *satisfying* 4.5.2)*, and assume that*

4.7.1) *there exist* $\alpha \in R$*,* $\alpha \geq 0$*, and a gauge* $\widehat{\gamma}$*, such that, for every* $\widehat{\gamma}$*-fine* k*-partition* Π *of* (a,b)*, we have:*

$$\left|\sum_{\Pi} U_n - (GH_k)\int_a^b U_n\right| \leq \alpha \quad \textit{for all}\, n \in \mathbb{N}.$$

Then U_0 *is* (GH_k) *integrable on* (a,b)*, and*

$$(GH_k)\int_a^b U_0 = (D)\lim_n (GH_k)\int_a^b U_n.$$

Remark 4.8 Condition 4.7.1) is analogous to property $H3$) introduced in Theorem 4.2.

Proof of Theorem 4.7: Since the sequence $\left((GH_k)\int_a^b U_n\right)_n$ is bounded and increasing, it admits the (D)-limit in R. Thus there is a (D)-sequence $(c_{i,j})_{i,j}$ in R such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $h_0 \in \mathbb{N}$ such that, $\forall\, h, l \geq h_0$,

$$\left|(GH_k)\int_a^b U_h - (GH_k)\int_a^b U_l\right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{15}$$

Furthermore, from 4.5.2) we get the existence of an element $w \in \mathbb{R}^+$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge γ^* such that, for every γ^*-fine k-partition Π of (a,b), $\Pi = \{(t_i; x_{i,1}, \ldots, x_{i,k-1}), [x_{i,0}, x_{i,k}] : i = 1, \ldots, q\}$, we have:

$$\begin{aligned} &\sum_{i=1}^{q} |U_0(t_i; x_{i,1}, \ldots, x_{i,k}) - U_{p(t_i)}(t_i; x_{i,1}, \ldots, x_{i,k})| \\ \leq\ & \sum_{i=0}^{q} h^*(t_i; x_{i,1}, \ldots, x_{i,k}) \left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)}\right) \leq w\left(\bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)}\right). \end{aligned} \tag{16}$$

Note that the natural numbers $p(t_i)$ (using the same notations as in formula (12) of condition 4.5.2)) can be chosen greater than h_0, where h_0 is related to the same φ chosen in (16). Since U_h is integrable for all $h \in \mathbb{N}$, then for each $h \in \mathbb{N}$ there exists a (D)-sequence $(a_{i,j}^{(h)})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is a gauge γ_h such that for every γ_h-fine k-partition Π we have

$$\left|\sum_{\Pi} U_h - (GH_k) \int_a^b U_h\right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i+h+1)}^{(h)}. \tag{17}$$

For each $i, j \in \mathbb{N}$, put $b_{i,j}^{(1)} = 4\, w\, a_{i,j}^*$ and $b_{i,j}^{(m)} = a_{i,j}^{(m-1)}$ $(m = 2, 3, \ldots)$. Moreover, let α be as in 4.7.1). By virtue of the Fremlin lemma 2.4 there exists a (D)-sequence $(b_{i,j})_{i,j}$ such that, for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $s \in \mathbb{N}$,

$$\alpha \wedge \left(\sum_{m=1}^{s}\left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(m)}\right)\right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}. \tag{18}$$

Let $\varphi \in \mathbb{N}^{\mathbb{N}}$, $h_0 = h_0(\varphi)$ be as in (15) and $\gamma_0(t) = \gamma^*(t) \cap \widehat{\gamma}(t) \cap \gamma_1(t) \cap \ldots \cap \gamma_{p(t)}(t)$, where the involved gauges are the ones associated with φ, as above. Choose any γ_0-fine k-partition $\Pi = \{(t_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, q\}$. Fix arbitrarily $h > h_0$, where h_0 is as in (15). We have:

$$\begin{aligned}
&\left|\sum_{\Pi} U_h - (GH_k) \int_a^b U_h\right| \qquad\qquad (19)\\
\leq\ &\left|\sum_{p(t_i)\geq h} [U_h(t_i; x_{i,1}, \ldots, x_{i,k}) - U_h(t_i; x_{i,0}, \ldots, x_{i,k-1})] - \sum_{p(t_i)\geq h} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_h\right| \\
+\ &\left|\sum_{p(t_i)<h} [U_h(t_i; x_{i,1}, \ldots, x_{i,k}) - U_h(t_i; x_{i,0}, \ldots, x_{i,k-1})] - \sum_{p(t_i)<h} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_h\right|.
\end{aligned}$$

Let $\widetilde{\Pi} = \{(t_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], h \leq p(t_i)\} \cup \left(\bigcup_{p(t_i)<h} \Pi_i\right)$, where Π_i is a sufficiently fine k-partition of $[x_{i,0}, x_{i,k}]$, in such a way that $\widetilde{\Pi}$ is a γ_h-fine k-partition of (a, b). Then

$$\left|\sum_{\widetilde{\Pi}} U_h - (GH_k) \int_a^b U_h\right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i+h+1)}^{(h)}.$$

Hence, by the Saks-Henstock lemma, we obtain

$$\left| \sum_{p(t_i)\geq h} [U_h(t_i; x_{i,1}, \ldots, x_{i,k}) - U_h(t_i; x_{i,0}, \ldots, x_{i,k-1})] - \sum_{p(t_i)\geq h} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_h \right|$$
$$\leq \bigvee_{i=1}^{\infty} a^{(h)}_{i,\varphi(i+h+1)}. \tag{20}$$

We now estimate the second part of the right side of (19). We have:

$$
\begin{aligned}
&\left| \sum_{p(t_i)<h} [U_h(t_i; x_{i,1}, \dots, x_{i,k}) - U_h(t_i; x_{i,0}, \dots, x_{i,k-1})] - \sum_{p(t_i)<h} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_h \right| \\
\leq & \left| \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} [U_h(t_i; x_{i,1}, \dots, x_{i,k}) - U_h(t_i; x_{i,0}, \dots, x_{i,k-1})] \right. \\
- & \left. \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} [U_m(t_i; x_{i,1}, \dots, x_{i,k}) - U_m(t_i; x_{i,0}, \dots, x_{i,k-1})] \right| + \\
+ & \left| \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} [U_m(t_i; x_{i,1}, \dots, x_{i,k}) - U_m(t_i; x_{i,0}, \dots, x_{i,k-1})] \right. \\
- & \left. \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_m \right| + \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{x_{i,0}}^{x_{i,k}} (U_h - U_m) \\
\leq & \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} |[U_h(t_i; x_{i,1}, \dots, x_{i,k}) - U_h(t_i; x_{i,0}, \dots, x_{i,k-1})] \qquad (21) \\
- & [U_m(t_i; x_{i,1}, \dots, x_{i,k}) - U_m(t_i; x_{i,0}, \dots, x_{i,k-1})]| \\
+ & \sum_{m=h_0}^{h-1} \left| \sum_{p(t_i)=m} [U_m(t_i; x_{i,1}, \dots, x_{i,k}) - U_m(t_i; x_{i,0}, \dots, x_{i,k-1})] \right. \\
- & \left. \sum_{p(t_i)=m} (GH_k) \int_{x_{i,0}}^{x_{i,k}} U_m \right| + \sum_{m=h_0}^{h-1} \sum_{p(t_i)=m} (GH_k) \int_{x_{i,0}}^{x_{i,k}} (U_h - U_m) \\
\leq & \bigvee_{i=1}^{\infty} b^{(1)}_{i,\varphi(i+1)} + \sum_{m=h_0}^{h-1} \bigvee_{i=1}^{\infty} a^{(m)}_{i,\varphi(i+m+1)} + (GH_k) \int_a^b (U_h - U_{h_0}) \\
\leq & \bigvee_{i=1}^{\infty} b^{(1)}_{i,\varphi(i+1)} + \sum_{m=2}^{h} \bigvee_{i=1}^{\infty} b^{(m)}_{i,\varphi(i+m)} + (GH_k) \int_a^b (U_h - U_{h_0}) \\
= & \sum_{m=1}^{h} \left(\bigvee_{i=1}^{\infty} b^{(m)}_{i,\varphi(i+m)} \right) + (GH_k) \int_a^b (U_h - U_{h_0}).
\end{aligned}
$$

Thus, from 4.5.2), (15), (18) and (21) we get the existence of a (D)-sequence $(d_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exist a gauge γ_0 and $h_0 \in \mathbb{N}$ such that, for each

γ_0-fine k-partition Π and for all $h > h_0$, we have:

$$\left| \sum_{\Pi} U_h - (GH_k) \int_a^b U_h \right| \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}. \tag{22}$$

The assertion follows from Theorem 4.5. □

Finally we prove a version of the Lebesgue dominated convergence theorem.

Theorem 4.9 *Let* $(U_n : (a,b)^{k+1} \to R)_n$ *be a sequence of* GH_k *integrable functions such that* $\bigvee_{n \in P_1, m \in P_2} |U_n - U_m|$ *is* GH_k *integrable for every* P_1, $P_2 \subset \mathbb{N}$, *and assume that* $U_0 : (a,b)^{k+1} \to R$, $h^* : (a,b)^{k+1} \to \mathbb{R}^+$ *are two maps, such that* 4.5.2) *holds. Then* U_0 *is* GH_k *integrable and*

$$(GH_k) \int_a^b U_0 = (D) \lim_n (GH_k) \int_a^b U_n.$$

Proof: For all $s \in \mathbb{N}$ and $h \geq s$, put

$$g_{s,h} = \bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m|;$$

moreover, for each $s \in \mathbb{N}$, set

$$g_s = \bigvee_{n,m \geq s} |U_n - U_m|.$$

We shall prove that, for each fixed $s \in \mathbb{N}$, the sequence $(g_{s,h})_{h \geq s}$ satisfies the hypothesis of Theorem 4.7.

First of all, it is easy to check that the sequence

$$\left((GH_k) \int_a^b g_{s,h} \right)_h$$

is well-defined and bounded in R.

Fix arbitrarily $s \in \mathbb{N}$. We have:

$$\begin{aligned} \bigvee_{n,m \geq s} |U_n - U_m| &= \left(\bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m| \right) \bigvee \left(\bigvee_{n,m \geq h} |U_n - U_m| \right) \\ &\leq \left(\bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m| \right) + \left(\bigvee_{n,m \geq h} |U_n - U_m| \right), \end{aligned}$$

and hence

$$0 \leq g_s - g_{s,h} \leq \bigvee_{n,m \geq h} |U_n - U_m| \quad \text{for all}\, h \geq s.$$

Since $(U_n)_n$ verifies 4.5.2), then the sequence $(g_{s,h})_h$ satisfies 4.5.2) too, with h^* as in our hypotheses and where the role of the "limit function" is played by g_s.

We now turn to 4.7.1). As $\bigvee_{n,m \in \mathbb{N}} |U_n - U_m|$ is GH_k integrable, there exist a gauge $\widehat{\gamma}$ and a positive element $a^* \in R$ such that, for every $\widehat{\gamma}$-fine k-partition

$$\Pi := \{(t_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, q\},$$

for all $s \in \mathbb{N}$ and $h \geq s$, we get:

$$\sum_{i=1}^{q} \left[\bigvee_{s \leq \min(n,m) \leq h} |U_n(t_i; x_{i,1}, \ldots, x_{i,k}) - U_m(t_i; x_{i,1}, \ldots, x_{i,k})| \right] \leq a^*, \quad (23)$$

that is

$$\sum_{i=1}^{q} g_{s,h}(t_i; x_{i,1}, \ldots, x_{i,k}) \leq a^*.$$

From this it follows that 4.7.1) is satisfied. Thus we get that, for every $s \in \mathbb{N}$, g_s is GH_k integrable and

$$\int_a^b g_s = \bigvee_{h \geq s} (GH_k) \int_a^b g_{s,h}.$$

We now prove that the sequence $(-g_s)_s$ satisfies the hypotheses of Theorem 4.7.

First of all, it is easy to check that the sequence $\left((GH_k) \int_a^b g_s \right)_s$ is bounded. Moreover, since

$$g_s = |-g_s| = \bigvee_{n,m \geq s} |U_n - U_m|$$

and $(U_n)_n$ satisfies 4.5.2), then the sequence $(-g_s)_s$ verifies 4.5.2) too, with h^* as in our hypotheses and where the role of the "limit function" is played by the null function.

Concerning 4.7.1), it is enough to check that the argument in (23) works even if we replace $\bigvee_{s \leq \min(n,m) \leq h} |U_n - U_m|$ with $\bigvee_{n,m \geq s} |U_n - U_m|$. Thus, we get

$$(D) \lim_s (GH_k) \int_a^b g_s = \bigwedge_{s \in \mathbb{N}} (GH_k) \int_a^b g_s = 0. \quad (24)$$

Proceeding analogously as in the proof of Theorem 4.7, it is possible to prove the existence of (D)-sequences $(e^{(m)}_{i,j})_{i,j}$, $m \in \mathbb{N}$, such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge γ' and $h' \in \mathbb{N}$ such that, for each γ'-fine k-partition $\Pi := \{(t_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, q\}$ and for all $h > h'$, we have:

$$\begin{aligned} \left| \sum_{\Pi} U_h - (GH_k) \int_a^b U_h \right| &\leq \sum_{m=1}^{h} \left(\bigvee_{i=1}^{\infty} e^{(m)}_{i,\varphi(i+m)} \right) + \sum_{m=h'}^{h-1} \sum_{p(t_i)=m} \left| (GH_k) \int_{x_{i,0}}^{x_{i,k}} (U_h - U_m) \right| \\ &\leq \sum_{m=1}^{h} \left(\bigvee_{i=1}^{\infty} e^{(m)}_{i,\varphi(i+m)} \right) + (GH_k) \int_a^b g_{h'}. \end{aligned} \tag{25}$$

From (25) we get the existence of a (D)-sequence $(d'_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a gauge γ' and $h' \in \mathbb{N}$ such that for each γ'-fine partition Π and $h > h'$, we have:

$$\left| \sum_{\Pi} U_h - (GH_k) \int_a^b U_h \right| \leq \bigvee_{i=1}^{\infty} d'_{i,\varphi(i)}. \tag{26}$$

The assertion follows from (26) and Theorem 4.5. □

5 Applications to Differential Calculus

We begin with introducing some concepts of variation. From now on we suppose that R is a weakly σ-distributive *Riesz commutative algebra,* that is a weakly σ-distributive Dedekind complete Riesz space endowed with a commutative "product" $\cdot : R \times R \to R$, compatible with the structures of sum, order, suprema and infima. Moreover, we assume that $a, b \in \mathbb{R}$ and E is a nonempty subset of $[a, b]$.
Given $f : [a, b] \to R$, $G : [a, b]^k \to R$, we call U or $U_{f,G}$ the function $U : [a, b]^{k+1} \to R$ defined by setting

$$U(\tau; t_1, \ldots, t_k) = f(\tau)\, G(t_1, \ldots, t_k), \quad \tau, t_1, \ldots, t_k \in [a, b].$$

If $U = U_{f,G} \in GH_k[a, b]$, we denote with $(GH_k) \int_a^b f\, dG$ and call also *generalized Henstock-Stieltjes integral* of f with respect to G the integral $(GH_k) \int_a^b U$.

Definition 5.1 Let $G : [a, b]^k \to R$, and fix a function $\delta : [a, b] \to \mathbb{R}^+$. For every δ-fine k-decomposition

$$\Pi := \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\}$$

of $[a, b]$, define

$$V_k(G, E, \delta, \Pi) = \sum_{\xi_i \in E} |G(x_{i,1}, \ldots, x_{i,k}) - G(x_{i,0}, \ldots, x_{i,k-1})|.$$

If there exist a map $\delta \in (\mathbb{R}^+)^{[a,b]}$ and an element $M \in R$, $M \geq 0$, such that $V_k(G, E, \delta, \Pi) \leq M$ for every δ-fine k-decomposition Π, we say that G is *k-variationally bounded* on E, in symbols $G \in BV_k(E)$.

We now state the following generalization of derivative.

Definition 5.2 Let $F, f : [a, b] \to R$, $G : [a, b]^k \to R$. We say that f is the *global k-derivative* of F with respect to G, and we write in symbols $f = \dfrac{dF}{dG}$, if there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a gauge $\delta = \delta(\varphi)$ such that

$$\begin{aligned} & |F(t_k) - F(t_0) - f(x) \cdot [G(t_1, \ldots, t_k) - G(t_0, \ldots, t_{k-1})]| \\ \leq \; & |G(t_1, \ldots, t_k) - G(t_0, \ldots, t_{k-1})| \cdot \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) \end{aligned}$$

whenever $a \leq t_0 \leq \ldots \leq t_k \leq b$, $x \in [t_0, t_k] \subset (x - \delta(x), x + \delta(x)) \subset [a, b]$.

We now turn to the following Fundamental Formula of Calculus.

Theorem 5.3 *If $G \in BV_k[a, b]$ and $f = \dfrac{dF}{dG}$, then $U_{f,G} \in GH_k[a, b]$ and*

$$(GH_k) \int_a^b f \, dG = F(b) - F(a).$$

Proof: Since $G \in BV_k[a, b]$, there exist a map $\delta_1 : [a, b] \to \mathbb{R}^+$ and an element $M \in R$, $M \geq 0$, such that $V_k(G, [a, b], \delta_1, \Pi) \leq M$ for every δ-fine k-decomposition Π of $[a, b]$, $\Pi := \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, \ldots, n\}$. Since $f = \dfrac{dF}{dG}$, there is a (D)-sequence $(a_{i,j})_{i,j}$ such that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a map $\delta : [a,b] \to \mathbb{R}^+$ such that $\delta(x) \le \delta_1(x)$ for all $x \in [a,b]$, and

$$\begin{aligned} & |F(t_k) - F(t_0) - f(x) \cdot [G(t_1,\ldots,t_k) - G(t_0,\ldots,t_{k-1})]| \\ \le\ & |G(t_1,\ldots,t_k) - G(t_0,\ldots,t_{k-1})| \cdot \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) \end{aligned}$$

whenever $a \le t_0 \le \ldots \le t_k \le b$, $x \in [t_0,t_k] \subset (x-\delta(x), x+\delta(x)) \subset [a,b]$.

Pick now any arbitrary δ-fine k-partition Π of (a,b),

$$\Pi := \{(\zeta_i; z_{i,1},\ldots,z_{i,k-1}) : [z_{i,0}, z_{i,k}], i = 1,\ldots,n\}.$$

Then we get:

$$\begin{aligned} & |F(b) - F(a) - \sum_{i=1}^{n} f(\zeta_i)\,[G(z_{i,1},\ldots,z_{i,k}) - G(z_{i,0},\ldots,z_{i,k-1})]| \\ \le\ & \sum_{i=1}^{n} |F(z_{i,k}) - F(z_{i,0}) - f(\zeta_i)\,[G(z_{i,1},\ldots,z_{i,k}) - G(z_{i,0},\ldots,z_{i,k-1})]| \\ \le\ & \left[\sum_{i=1}^{n} |G(z_{i,1},\ldots,z_{i,k}) - G(z_{i,0},\ldots,z_{i,k-1})| \right] \cdot \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) \\ \le\ & M \cdot \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right). \end{aligned}$$

From this it follows that $U_{f,G} \in GH_k[a,b]$ and

$$(GH_k) \int_a^b f\,dG = F(b) - F(a). \quad \square$$

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