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Fatou's Lemma for Multifunctions with Unbounded Values in a Dual Space

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A version of Fatou's lemma for multifunctions with unbounded values in infinite dimensions is presented. It generalizes both the recent Fatou-type results for Gelfand integrable functions of Cornet-Martins da Rocha [18] and, in the case of finite dimensions, the finite-dimensional version of the unifying multivalued Fatou-type result of Balder-Hess [12].

Keywords: Fatou's lemma in everal dimensions, Gelfand integral, Young measure, asymptotic cone

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1. Introduction

Fatou's lemma in finitely many dimensions goes back to Aumann [2] and Schmeidler [32]. It plays an important technical role in the usual proofs of competitive equilibrium existence. Related versions of Fatou's lemma were given by Artstein and Hildenbrand-Mertens [1, 24], and in [3] a version was given that subsumes the aforementioned ones. In another development, Olech introduced the use of cones of directions with uniform integrability properties [30]. Extensions of Fatou's lemma to infinite dimensions were given by Khan-Majumdar [26] and Yannelis [33], in [5] and by Castaing-Clauzure [14]; such extensions are usually of an approximate nature because of the failure of Lyapunov's theorem to hold in infinite dimensions. Multivalued versions of Fatou's lemma were given by Pucci-Vitillaro [31], Hiai [23], Hess [22] and by Balder-Hess [12]. The results in [12] are of a quite general and unifying nature. They are stated in two somewhat different versions, i.e., finite- and infinite-dimensional ones, and apply to multifunctions that can have unbounded values. As is explained in [12], those results contain all the aforegoing results, including the single-valued ones (with the exception of [14] – see [13] for more on that type of result). All the aforementioned extensions to infinite dimensions involve *Bochner integrable* functions. Motivated by general equilibrium existence questions in a model of spatial economies [16], Cornet-Médécin [17] gave a Fatou lemma for *Gelfand integrable* (also called weak star integrable) functions that map into the dual of an infinite-dimensional Banach space. Their

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result was improved in [11]. Subsequently, that improvement was again sharpened by Cornet-Martins da Rocha [18]. In their paper it is shown that, unlike the current situation for Bochner integrable functions, an infinite-dimensional result can be formulated for Gelfand integrable functions that does not require a separate and parallel development in finite dimensions. In other words, it can be applied to finite dimensions without any loss of power. Thus, the result in [18] includes the unifying finite-dimensional Fatou lemma of Balder [3]. The present paper continues this development. It presents a multivalued Fatou-type result that generalizes not only Theorems 2.1, 2.2 and Corollary 2.1 of [18], but also the finite-dimensional version of the multivalued Fatou lemmas of Balder-Hess, as given in [12, Theorem 3.2].

2. Preliminaries

Let $(X, \|\cdot\|)$ be a separable Banach space; on X we consider the norm topology. Let $Y = X^*$ be the dual of X, endowed with the w^*-topology $\sigma(Y, X)$. We use the usual symbols w^*-cl, w^*-seq-cl and co to denote the w^*-closure, the w^*-sequential closure and the convex hull of a subset of Y respectively (recall that the sequential closure of a set is the intersection of all sequentially closed sets containing that set). The dual norm on Y is given by $\|y\|^* = \sup_{x\in X, \|x\|\le 1} |\langle x, y\rangle|$, which also shows that $y \mapsto \|y\|^*$ is lower semicontinuous for the w^*-topology. The radius of a bounded set K of Y is denoted by $\|K\|^* = \sup_{y\in K} \|y\|^*$. We also denote $\bar{B}^*(0;\rho) := \{y \in Y : \|y\|^* \le \rho\}$ and $B^*(0;\rho) := \{y \in Y : \|y\|^* < \rho\}$ for $\rho > 0$; both sets are w^*-metrizable, and the former set is also w^*-compact by the Alaoglu-Bourbaki theorem. So $Y = \cup_{n\in\mathbb{N}} \bar{B}^*(0;n)$ is certainly a Suslin space. Hence, the Borel σ-algebra $\mathcal{B}(Y)$ on Y is the same for the w^*- and the dual norm topology (this can also be demonstrated directly, of course). For $y \in Y$ recall that the *Dirac probability measure* concentrated at y is the probability measure ϵ_y on $(Y, \mathcal{B}(Y))$, defined by $\epsilon_y(B) := 1$ if $B \ni y$ and $\epsilon_y(B) := 0$ otherwise.

For every nonempty $V \subset Y$ the *support function* of V is the functional $s(\cdot \mid V) : X \to (-\infty, +\infty]$ defined by $s(x \mid V) := \sup_{y\in V}\langle x, y\rangle$. The *asymptotic cone* of a set $V \subset Y$, denoted by As(V), is defined by

$$\mathrm{As}(V) := \{y \in Y : \langle x, y\rangle \le 0 \text{ for every } x \in X \text{ with } s(x \mid V) < +\infty\}.$$

Thus, As(V) is precisely the negative polar cone of the effective domain of the support function $s(\cdot \mid V)$, for we recall that the *negative polar cone* C^* of a cone $C \subset X$ is the set of all $y \in Y$ such that $\sup_{x\in C}\langle x, y\rangle \le 0$. Evidently, the asymptotic cone of V coincides with the asymptotic cone of the closed convex hull cl co V of V, and when V itself is closed and convex then As(V) is also the asymptotic cone in the classical sense of convex analysis (apply [27, 6.8.5]). We adopt the following unifying device; it was introduced by Cornet-Martins da Rocha [18] and will enable us to treat *implicitly* the situation where X and Y are finite-dimensional. Let $\mathcal{H}$ the family of all finite subsets of X. For every $H \in \mathcal{H}$ we define

$$H^\perp := \{y \in Y : \langle x, y\rangle = 0 \text{ for all } x \in H\}.$$

Evidently, this gives for any $V \subset Y$

$$V \subset \bigcap_{H\in\mathcal{H}} (V + H^\perp) \subset w^*\text{-cl } V \tag{1}$$

and, in fact, the first inclusion becomes an identity when X is finite-dimensional. For any sequence $(V_k)_k$ of subsets $V_k \subset Y$ the *sequential w^*-limes superior* $w^*\text{-Ls}_k V_k$ of that sequence (in the sense of Kuratowski) is defined as the set of all $\bar{y} \in Y$ for which there exist a subsequence $(V_{k_j})_j$ of $(V_k)_k$ and a corresponding sequence $(y_{k_j})_j$ in Y, with $y_{k_j} \in V_{k_j}$ for every $j \in \mathbb{N}$, such that $\bar{y} = w^*\text{-lim}_j\, y_{k_j}$. The space Y is in general non-metrizable for the w^*-topology, so the sequential limes superior set $w^*\text{-Ls}_k V_k$ does not have to be w^*-closed. However, it is certainly *sequentially* w^*-closed:

Proposition 2.1. *For every sequence $(V_k)_k$ of subsets $V_k \subset Y$ the sequential w^*-limes superior set $w^*\text{-Ls}_k V_k$ is sequentially w^*-closed.*

Proof. Let $(\bar{y}_n)_n$ be a sequence in $w^*\text{-Ls}_k V_k$ that w^*-converges to $\bar{y} \in Y$. By the Banach-Steinhaus theorem $(\bar{y}_n)_n$ is bounded for the dual norm, whence contained in $B^*(0;\rho)$ for some $\rho > 0$. As observed earlier, such a ball is w^*-metrizable; let d^* stand for some associated metric. Then there exists $\bar{y}_{n_1}$ in $B^*(0;\rho)$ with $d^*(\bar{y}_{n_1}, \bar{y}) < 1/2$; hence there also exist a set V_{k_1} and $y_{k_1} \in B^*(0;\rho) \cap V_{k_1}$ such that $d^*(\bar{y}_{n_1}, y_{k_1}) < 1/2$. So $d^*(\bar{y}, y_{k_1}) < 1$. Similarly, there exists $\bar{y}_{n_2} \in B^*(0;\rho)$ with $d^*(\bar{y}_{n_2}, \bar{y}) < 1/4$; hence there exist a set V_{k_2}, $k_2 > k_1$, and $y_{k_2} \in B^*(0;\rho) \cap V_{k_2}$ such that $d^*(\bar{y}_{n_2}, y_{k_2}) < 1/4$. So $d^*(\bar{y}, y_{k_2}) < 1/2$, etc. In this way a sequence $(y_{k_p})_p$ is obtained with $y_{k_p} \in V_{k_p}$ and $d^*(\bar{y}, y_{k_p}) < 2^{1-p}$. □

Let $(\Omega, \mathcal{A}, \mu)$ be a complete[1] finite measure space. As is well-known [20], Ω can be partitioned as $\Omega = \Omega^{pa} \cup (\Omega \backslash \Omega^{pa})$, where Ω^{pa} is the purely atomic part of the measure space $(\Omega, \mathcal{A}, \mu)$, i.e., the union of all its non-null atoms, of which there are at most countably many. Then $\Omega^{na} := \Omega \backslash \Omega^{pa}$, equipped with the traces of $\mathcal{A}$ and μ, forms a nonatomic measure space. As is usual, we denote the (prequotient) space of all μ-integrable real-valued functions on Ω by $\mathcal{L}^1_{\mathbb{R}}(\Omega)$. Recall that a sequence $(\phi_k)_k \in \mathcal{L}^1_{\mathbb{R}}(\Omega)$ is said to be *uniformly integrable* if

$$\lim_{a\to\infty} \sup_k \int_{\{|\phi_k| \geq a\}} |\phi_k|\, d\mu = 0.$$

The *outer integral* of a function $\psi : \Omega \to [-\infty, +\infty]$ is defined by

$$\int^*_\Omega \psi\, d\mu := \inf\{\int_\Omega \phi\, d\mu : \ \phi \in \mathcal{L}^1_{\mathbb{R}}(\Omega),\ \phi \geq \psi \text{ a.e. in } \Omega\},$$

where the infimum is defined to be $+\infty$ when the set is empty. Of course, one easily observes that when ψ itself is in $\mathcal{L}^1_{\mathbb{R}}(\Omega)$, then the above outer integral coincides with the classical integral. A function $f : \Omega \to Y$ is said to be *Gelfand integrable* if it is X-scalarly integrable, i.e., if the scalar function $\langle x, f\rangle : \omega \mapsto \langle x, f(\omega)\rangle$ belongs to $\mathcal{L}^1_{\mathbb{R}}(\Omega)$ for every $x \in X$. It is convenient to denote the collection of all Gelfand integrable functions by $\mathcal{L}^1_Y(\Omega)[X]$. For every $f \in \mathcal{L}^1_Y(\Omega)[X]$ the following property holds automatically (see [19, p. 52] or [17, footnote 2]): for every $A \in \mathcal{A}$ there exists a unique element y_A in Y, called the *Gelfand integral* or *w^*-integral* of f over A and denoted by $y_A = \int_A f\, d\mu$, such that

$$\langle x, y_A\rangle = \int_A \langle x, f\rangle\, d\mu \text{ for every } x \in X.$$

[1]Completeness is not really needed, but it facilitates the proofs. Because our main results hold modulo null sets, it can be dropped in the usual way (first complete the measure space and select a.e.-equivalent modifications afterwards).

The *Gelfand integral* of a multifunction $F : \Omega \to 2^Y$ is defined in the sense of Aumann [2]. That is, we define it to be the subset $\int_\Omega F\, d\mu$ of Y which is given by

$$\int_\Omega F\, d\mu := \{\int_\Omega f\, d\mu : f \in \mathcal{L}^1_Y(\Omega)[X],\ f(\omega) \in F(\omega) \text{ for a.e. } \omega \text{ in } \Omega\}.$$

Because we allow this set to be empty, this definition does not require any measurability properties for F.

3. Main results

Our main result is a Fatou-type lower closure result for the Gelfand integrals of a sequence $(F_k)_k$ of multifunctions $F_k : \Omega \to 2^Y$. We shall use the same structural assumptions as in [12]. Observe that in principle no measurability is required for the multifunctions F_k. Let L be a subset of Y whose closed convex hull is locally w^*-compact and does not contain any line. We suppose that

(A_1) $F_k(\omega) \subset G_k(\omega) + r_k(\omega)L$ for every $k \in \mathbb{N}$ and a.e. ω in Ω,

where $(r_k)_k \subset \mathcal{L}^1_{\mathbb{R}}(\Omega)$ is a uniformly integrable sequence and $(G_k)_k$ is a sequence of w^*-compact-valued multifunctions $G_k : \Omega \to 2^Y$ such that

(A_2) $\sup_k \int^*_\Omega \|G_k(\omega)\|^*\ \mu(d\omega) < +\infty$.

Here outer integration is used, as introduced in Section 2; this avoids unnecessary measurability considerations for the G_k. Consider the following cone in X:

$$C_{00} := \{x \in X : (\max\{0, s(-x \mid G_k)\})_k \text{ is uniformly integrably bounded}\}.$$

Here the sequence $(\psi_k)_k$ of nonnegative functions $\psi_k : \omega \mapsto \max\{0, s(-x \mid G_k(\omega))\}$ is said to be *uniformly integrably bounded* if there exists a uniformly integrable sequence $(\phi_k)_k$ in $\mathcal{L}^1_{\mathbb{R}}(\Omega)$ such that $0 \le \psi_k \le \phi_k$ a.e. for all k.

Theorem 3.1. *Under assumptions* (A_1)-(A_2)

$$w^*\text{-Ls}_k \int_\Omega F_k\, d\mu \subset \bigcap_{H\in\mathcal{H}} (\int_\Omega F_{00}\, d\mu + \text{As}(L - C^*_{00}) + H^\perp) \subset w^*\text{-cl}\,(\int_\Omega F_{00}\, d\mu + \text{As}(L - C^*_{00}))$$

and

$$w^*\text{-Ls}_k \int_\Omega F_k\, d\mu \subset \int_{\Omega^{pa}} F_{00}\, d\mu + \int_{\Omega^{na}} w^*\text{-cl co } F_{00}\, d\mu + \text{As}(L - C^*_{00}),$$

where the multifunction $F_{00} : \Omega \to 2^Y$ *is defined by* $F_{00}(\omega) := w^*\text{-Ls}_k F_k(\omega)$, $\omega \in \Omega$.

The proof of this result is given in Section 4. In view of what was observed about (1), an immediate consequence of Theorem 3.1 is the following corollary, which is the principal finite-dimensional result of [12] (the infinite-dimensional version of this result in [12], which is its Theorem 3.1, is for Bochner integrable functions; hence, it is not directly comparable to the results presented here).

Corollary 3.2 ([12, Theorem 3.2]). *Suppose that X is finite dimensional. Under assumptions* (A_1)-(A_2)

$$\text{Ls}_k \int_\Omega F_k\, d\mu \subset \int_\Omega F_{00}\, d\mu + \text{As}(L - C^*_{00})$$

and

$$\mathrm{Ls}_k \int_\Omega F_k \, d\mu \subset \int_{\Omega^{pa}} F_{00} \, d\mu + \int_{\Omega^{na}} \mathrm{cl\ co}\ F_{00} \, d\mu + \mathrm{As}(L - C_{00}^*).$$

Of course, in this case we have $F_{00}(\omega) := \mathrm{Ls}_k F_k(\omega)$, and the condition for L comes down to requiring that the closed convex hull of L does not contain any line. Example 3.7 in [12] demonstrates that the latter condition is indispensable; this simple example uses the Lebesgue unit interval as the underlying measure space $(\Omega, \mathcal{A}, \mu)$, employs $X = Y = L = \mathbb{R}$ and has $F_k \equiv \{k\}$ on $[0, 1/2]$ and $F_k \equiv \{-k\}$ on $(1/2, 1]$.

We now specialize Theorem 3.1 to the single-valued case $F_k := \{f_k\}$, $G_k := \{g_k\}$, where $(f_k)_k$ and $(g_k)_k$ are given sequences in $\mathcal{L}^1_Y(\Omega)[X]$. Then the previous cone C_{00} specializes into the following cone C_0:

$$C_0 := \{x \in X : (\max\{0, -\langle x, f_k\rangle\})_k \text{ is uniformly integrable}\}.$$

We formulate the following assumptions: We suppose that

(A_1') $f_k(\omega) \in g_k(\omega) + r_k(\omega) L$ for every $k \in \mathbb{N}$ and a.e. ω in Ω,

where $(r_k)_k \subset \mathcal{L}^1_{\mathbb{R}}(\Omega)$ is a uniformly integrable sequence and $(g_k)_k$ is a sequence such that

(A_2') $\sup_k \int_\Omega \|g_k(\omega)\|^* \, \mu(d\omega) < +\infty$.

Corollary 3.3. *Under assumptions* (A_1')-(A_2')

$$w^*\text{-}\mathrm{Ls}_k \int_\Omega f_k \, d\mu \subset \bigcap_{H \in \mathcal{H}} \Big(\int_\Omega F_0 \, d\mu + As(L - C_0^*) + H^\perp \Big) \subset w^*\text{-cl} \Big(\int_\Omega F_0 \, d\mu + \mathrm{As}(L - C_0^*) \Big)$$

and

$$w^*\text{-}\mathrm{Ls}_k \int_\Omega f_k \, d\mu \subset \int_{\Omega^{pa}} F_0 \, d\mu + \int_{\Omega^{na}} w^*\text{-cl co}\ F_0 \, d\mu + \mathrm{As}(L - C_0^*)),$$

where the multifunction $F_0 : \Omega \to 2^Y$ *is defined by* $F_0(\omega) := w^*\text{-}\mathrm{Ls}_k\{f_k(\omega)\}$, $\omega \in \Omega$.

The main Theorems 2.1 and 2.2 of Cornet-Martins da Rocha [18] follow from this result (take $L := \{0\}$, which causes $f_k = g_k$ – see Corollary 4.1 below), as does their Corollary 2.1: take L to be a pointed locally w^*-compact closed convex cone and set $r_k \equiv 1$. Still in infinite dimensions, Corollary 3.3 also generalizes the Fatou lemmas of Balder [11] and Cornet-Médécin [17]. In finite dimensions, Corollary 3.3 coincides with Corollary 4.3 of [12]. Consequently, Corollary 3.3 also generalizes the finite dimensional Fatou lemmas of Artstein [1], Aumann [2], Balder [3], Hildenbrand and Mertens [24], Olech [30] and Schmeidler [32].

4. Proofs.

The proof of Theorem 3.1 is based on an idea already pursued in [12], namely that a multivalued Fatou-type result can actually be obtained from its single-valued specialization, i.e., the following corollary, which is a further specialization of Corollary 3.3 for $L := \{0\}$. To formulate this specialization, we need only one assumption, namely:

(A') $\theta := \sup_k \int_\Omega \|f_k\|^* \, d\mu < +\infty$.

Corollary 4.1 ([18, Theorems 2.1,2.2]). *Under assumption (A')*

$$w^*\text{-Ls}_k \int_\Omega f_k \, d\mu \subset \bigcap_{H\in\mathcal{H}} (\int_\Omega F_0 \, d\mu - C_0^* + H^\perp) \subset w^*\text{-cl} \, (\int_\Omega F_0 \, d\mu - C_0^*) \tag{1}$$

and

$$w^*\text{-Ls}_k \int_\Omega f_k \, d\mu \subset \int_{\Omega^{pa}} F_0 \, d\mu + \int_{\Omega^{na}} w^*\text{-cl co } F_0 \, d\mu - C_0^*. \tag{2}$$

This result is proven in [18] by means of its Theorem 3.1, which is an infinite-dimensional extension of Komlos theorem that builds on its Proposition 4.1 (which is a special case of [7, Theorem 2.1]) and several other intermediate results. Here we shall provide a new, shorter proof, which is based on Young measure theory. It is therefore in line with the proofs given in [3, 5, 11]. We refer the reader to the introductory section of [5] for a broad outline of this approach. In fact, our proof of Corollary 4.1 considerably sharpens the Young measure based proof of [11, Theorems 1.1, 1.2]. The improvements stem from the use of Proposition 2.1 and a result from [8] (incidentally, the resulting proof shows quite some similarity with the proof of the finite-dimensional Fatou lemma in [8, Proposition 3.5]). Thereupon we shall use Corollary 4.1 to prove Theorem 3.1. Just as in [12], the key instrument for this is Lemma 4.8, a result due to Hess [22].

Our proof of Corollary 4.1 will involve the next six lemmas. Throughout, the topology used on Y will be the w^*-topology. To begin with, we let $a \in w^*\text{-Ls}_k \int_\Omega f_k \, d\mu$ be fixed and arbitrary. We start with the preliminary selection of a suitable subsequence of $(f_k)_k$.

Lemma 4.2. *There exist a subsequence $(f_{k_j})_j$ of $(f_k)_k$ such that $a = w^*\text{-}\lim_j \int_\Omega f_{k_j} \, d\mu$ and such that the limit $\hat{f}(\omega) := w^*\text{-}\lim_j f_{k_j}(\omega)$ exists for a.e. ω in Ω^{pa}.*

Proof. By definition of the sequential limes superior set, there exists a subsequence $(f_{k_m})_m$ of $(f_k)_k$ such that $a = w^*\text{-}\lim_m \int_\Omega f_{k_m} \, d\mu$. Since each f_{k_m} is a.e. constant (say equal to $c_m \in Y$) on each non-null atom A_i of Ω, we have $\sup_m \|c_m\|^* \leq \theta/\mu(A_i)$ for every i, as a consequence of (A'). So, by w^*-compactness and metrizability of the ball $\bar{B}^*(0;\theta/\mu(A_i))$ in Y, we can apply a diagonal extraction argument to ensure the existence of a further subsequence $(f_{k_j})_j$ of $(f_{k_m})_m$ such that $\hat{f}(\omega) := w^*\text{-}\lim_j f_{k_j}(\omega)$ exists for a.e. ω in Ω^{pa}. □

Recall that a *Young measure* from Ω to Y is a transition probability δ with respect to $(\Omega, \mathcal{A})$ and $(Y, \mathcal{B}(Y))$; in other words, δ is a function from Ω into the probability measures on Y such that $\omega \mapsto \delta(\omega)(B)$ is $\mathcal{A}$-measurable for every $B \in \mathcal{B}(Y)$. The set of all Young measures from Ω into Y will be denoted by $\mathcal{Y}(\Omega; Y)$. It is equipped with the *narrow* topology, for which we refer to [9].

Lemma 4.3. *To the sequence $(f_{k_j})_j$ in Lemma 4.2 there correspond a further subsequence $(f_{k_p})_p$ and a Young measure $\delta \in \mathcal{Y}(\Omega; Y)$ such that the sequence $(\epsilon_{f_{k_p}})_p$ of Dirac transition probabilities $\epsilon_{f_{k_p}} : \omega \mapsto \epsilon_{f_{k_p}(\omega)}$ converges to δ in the narrow topology. Moreover, δ satisfies*

$$\int_\Omega [\int_Y \|y\|^* \, \delta(\omega)(dy)] \mu(d\omega) \leq \theta, \tag{3}$$

$$\delta(\omega)(F_0(\omega)) = 1 \text{ for a.e. } \omega \text{ in } \Omega, \tag{4}$$

$$\delta(\omega) = \epsilon_{\hat{f}}(\omega) \text{ for a.e. } \omega \in \Omega^{pa}. \tag{5}$$

This result is contained in the proof on pp. 322-326 of [11], but instead of (4) one only finds there $\delta(\omega)(w^*\text{-cl}F_0(\omega)) = 1$ a.e. (see formula (3.3) in [11]). The present form, which is more more precise, follows directly from the observation contained in Proposition 2.1. This improvement was instigated by [18], although that reference does not use Young measures.

Proof. By Prohorov's theorem for Young measures [6, 10], the existence of a narrowly convergent subsequence and its narrow limit δ is guaranteed if we can demonstrate that $(\epsilon_{f_{k_j}})_j$ is *tight* in the sense of [4] (here we also use the fact that Y is a completely regular Suslin space for the w^*-topology). By (A') the tightness follows, since $y \mapsto \|y\|^*$ is inf-compact on Y. The inequality (3) then follows, by (A') and narrow convergence combined, from the fact that $y \mapsto \|y\|^*$ is also lower semicontinuous (apply the portmanteau theorem for narrow convergence [9, Theorem 4.10]). Also, by the support theorem for narrow convergence [9, Theorem 4.15(ii)] we obtain

$$\delta(\omega)(w^*\text{-seq-cl}(w^*\text{-Ls}_p\{f_{k_p}(\omega)\})) = 1 \text{ for a.e. } \omega \text{ in } \Omega$$

for the narrow limit δ. Because of Lemma 4.2, this immediately implies (5) and because of $w^*\text{-seq-cl}(w^*\text{-Ls}_p\{f_{k_p}(\omega)\}) \subset w^*\text{-seq-cl}(w^*\text{-Ls}_k\{f_k(\omega)\})$, the above also implies (4), for it was already observed in Proposition 2.1 that the sequential limes superior sets $w^*\text{-Ls}_k\{f_k(\omega)\}$ are always sequentially w^*-closed. □

Lemma 4.4. *There exist $f \in \mathcal{L}^1_Y(\Omega)[X]$ and $b := \int_\Omega f\, d\mu \in Y$ such that*

$$\langle x, f(\omega)\rangle = \int_Y \langle x, y\rangle \delta(\omega)(dy) \text{ for all } x \in X \text{ for a.e. } \omega \text{ in } \Omega. \tag{6}$$

$$f(\omega) \in w^*\text{-cl co } F_0(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{na}, \tag{7}$$

$$f(\omega) = \hat{f}(\omega) \in F_0(\omega) \text{ for a.e. } \omega \text{ in } \Omega^{pa}, \tag{8}$$

$$b - a \in C_0^*. \tag{9}$$

Proof. The inequality $\int_Y \|y\|^* \delta(\omega)(dy) < +\infty$ holds for a.e. ω in Ω, as is immediate by (3). Every probability measure ν on $(Y, \mathcal{B}(Y))$ with $\int_Y \|y\|^* \nu(dy) < +\infty$ has a unique resultant (also called barycenter) $y_\nu \in Y$, for which $\langle x, y_\nu\rangle = \int_Y \langle x, y\rangle \nu(dy)$ for all $x \in X$. This follows either by Lemma 1 of [11] or by simply observing that $x \mapsto \int_Y \langle x, y\rangle \nu(dy)$ is norm-continuous on X [28]. So the above implies that for a.e. ω in Ω the probability measure $\delta(\omega)$ has a barycenter, which we shall denote by $f(\omega)$; this means that (6) holds for all such ω. Besides, on the exceptional null set we set $f(\omega) := 0$. Then scalar measurability of f follows, by (6), from standard measurability results for integration over transition probabilities [29, Section III.2]. Moreover, by (3) the same lemma in [11] implies that the function f is also scalarly integrable, that is to say, Gelfand integrable. Also, (7) follows directly from (4) by the Hahn-Banach theorem and (8) follows by (5). Now $b := \int_\Omega f\, d\mu \in Y$ is well-defined in the sense of Gelfand, which implies that for every $x \in X$

$$\langle x, b\rangle = \int_\Omega \langle x, f\rangle\, d\mu = \int_\Omega [\int_Y \langle x, y\rangle \delta(\omega)(dy)]\mu(d\omega).$$

For every $x \in C_0$ it follows from the narrow convergence established in Lemma 4.3 that

$$\langle x, a\rangle = \lim_p \int_\Omega \langle x, f_{k_p}\rangle\, d\mu \geq \int_\Omega [\int_Y \langle x, y\rangle \delta(\omega)(dy)]\mu(d\omega).$$

Namely, for $x \in C_0$ one can apply the Fatou-Vitali part (e) of Theorem 4.10 in [9] to the integrand $(\omega, y) \mapsto \langle x, y\rangle$. Because of the previous identity, this inequality proves (9). $\square$

In particular, (2) follows from Lemma 4.4 (write $a = \int_\Omega f \, d\mu + a - b$). To prove (1), let $H := \{x_1, \ldots, x_m\} \in \mathcal{H}$ be arbitrary. First, we prove two results about measurability that will play a role in the proof of the concluding Lemma 4.7:

Lemma 4.5. *The graph of the multifunction $F_0 : \Omega \to 2^Y$ is $\mathcal{A} \otimes \mathcal{B}(Y)$-measurable. Moreover, under assumption (A') one has $F_0(\omega) \neq \emptyset$ for a.e. ω in Ω*

Proof. Since a w^*-convergent sequence is always bounded in the dual norm, it follows immediately that $F_0(\omega) = \cup_{q=1}^\infty F_q(\omega)$ for every $\omega \in \Omega$, similarly to [21, Remark 3.4]. Here $F_q(\omega) := w^*\text{-Ls}_k(\{f_k(\omega)\} \cap \bar{B}^*(0;q))$. So it is enough to prove that the graph of $F_q : \Omega \to 2^Y$ is measurable. Now the ball $\bar{B}^*(0;q)$ is w^*-metrizable, so it is well-known that for every $\omega \in \Omega$ and $q \in \mathbb{N}$

$$F_q(\omega) = \cap_{m=1}^\infty w^*\text{-cl} \left(\{f_k(\omega) : k \geq m\} \cap \bar{B}^*(0;q)\right).$$

By X-scalar measurability of every function f_k, it follows from [15, Theorem III.36] that its graph is also $\mathcal{A} \otimes \mathcal{B}(Y)$-measurable (here we use the fact that Y is a Suslin space for the w^*-topology). Hence, also the intersection of that graph with the set $\Omega \times \bar{B}^*(0;q)$ is measurable. So the w^*-compact-valued multifunction $\omega \mapsto \{f_k(\omega)\} \cap \bar{B}^*(0;q)$ from Ω into $2^{\bar{B}^*(0;q)}$ is measurable for every k (apply [15, Theorem III.30]). Therefore, it follows by [15, Proposition III.4] that the multifunction F_q is measurable (here we again use w^*-metrizability of $\bar{B}^*(0;q)$). Finally, (A') implies $\int_\Omega \liminf_k \|f_k\|^* \, d\mu \leq \theta < +\infty$, by Fatou's lemma. So $\liminf_k \|f_k(\omega)\|^* < +\infty$ for almost every ω; by w^*-metrizability and w^*-compactness of the balls $\bar{B}^*(0;\rho)$, $\rho > 0$, this implies that $F_0(\omega)$ is nonempty for every non-exceptional ω. $\square$

We consider now the multifunction $\Gamma_0 : \Omega \to 2^{\mathbb{R}^{m+1}}$, given by

$$\Gamma_0(\omega) := \{(\langle x_1, y\rangle, \ldots, \langle x_m, y\rangle, \xi^{m+1}) \; : \; y \in F_0(\omega), \xi^{m+1} \in \mathbb{R}, \xi^{m+1} \geq \|y\|^*\}.$$

Lemma 4.6. *The multifunction $\Gamma_0 : \Omega \to 2^{\mathbb{R}^{m+1}}$ has closed values and a $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^{m+1})$-measurable graph.*

Proof. Fix $\omega \in \Omega$. First, we prove that $\Gamma_0(\omega)$ is closed. Let $(\xi_k)_k$ be a sequence in $\Gamma_0(\omega)$, with $\xi_k^i = \langle x_i, y_k\rangle$ for $i = 1, \ldots, m$ and $\xi_k^{m+1} \geq \|y_k\|^*$ for $y_k \in F_0(\omega)$. Suppose that $(\xi_k)_k$ converges to $\bar{\xi} \in \mathbb{R}^{m+1}$. Then evidently $\sup_k \|y_k\|^* < +\infty$, so by the fact that balls in Y are w^*-metrizable, we conclude that a subsequence of $(y_k)_k$ w^*-converges to some vector in Y, which must belong to $F_0(\omega)$ by Proposition 2.1. By w^*-continuity of $\langle x_i, \cdot\rangle$ and w^*-lower semicontinuity of the dual norm, it easily follows that $\bar{\xi}$ belongs to $\Gamma_0(\omega)$. Next, we observe that $\Gamma_0(\omega) = \cup_{q=1}^\infty \Gamma_q(\omega)$, where

$$\Gamma_q(\omega) := \{(\langle x_1, y\rangle, \ldots, \langle x_m, y\rangle, \xi^{m+1}) \; : \; y \in F_q(\omega), \xi^{m+1} \in \mathbb{R}, \xi^{m+1} \geq \|y\|^*\},$$

with $F_q(\omega) := F_0(\omega) \cap \bar{B}^*(0;q)$ as introduced in the proof of Lemma 4.5. So it is enough to prove measurability of the graph of Γ_q for an arbitrary $q \in \mathbb{N}$. To this end, let $E \subset \mathbb{R}^{m+1}$ be arbitrary and closed. Correspondingly, we define E' as the closed set of all

$(y, \xi^{m+1}) \in Y \times \mathbb{R}$ such that both $(\langle x_1, y\rangle, \ldots, \langle x_m, y\rangle, \xi^{m+1}) \in E$ and $\xi^{m+1} \geq \|y\|^*$. Then the easy identity

$$\{\omega \in \Omega : \Gamma_q(\omega) \cap E \neq \emptyset\} = \{\omega \in \Omega : (F_q(\omega) \times \mathbb{R}) \cap E' \neq \emptyset\}$$

describes a $\mathcal{A}$-measurable set, because the multifunction $F_q \times \mathbb{R} : \Omega \to 2^{\bar{B}^*(0;q)\times\mathbb{R}}$ is evidently measurable, in view of the proof of Lemma 4.5 (apply Proposition 2.3 of [25]). This proves that Γ_q is measurable. Because $\bar{B}^*(0; q)$ is compact and metrizable for the w^*-topology, application of [15, Theorem III.30] gives that the graph of Γ_q is $\mathcal{A}\otimes\mathcal{B}(\mathbb{R}^{m+1})$-measurable. □

Lemma 4.7. *There exists $f_H \in \mathcal{L}^1_Y(\Omega)[X]$ such that $f_H(\omega) \in F_0(\omega)$ for a.e. ω in Ω and $b \in \int_\Omega f_H \, d\mu + H^\perp$.*

Proof. Let $\Phi(y) := (\langle x_1, y\rangle, \ldots, \langle x_m, y\rangle, \|y\|^*)$, $y \in Y$; this defines a measurable mapping $\Phi : Y \to \mathbb{R}^{m+1}$. We define $\delta^\Phi(\omega)(B) := \delta(\omega)(\Phi^{-1}(B))$ for $\omega \in \Omega$ and $B \in \mathcal{B}(Y)$; this yields the transition probability δ^Φ in $\mathcal{Y}(\Omega; \mathbb{R}^{m+1})$. By (4) and the trivial inclusion $F_0(\omega) \subset \Phi^{-1}(\Gamma_0(\omega))$ it follows that

$$\delta^\Phi(\omega)(\Gamma_0(\omega)) = 1 \text{ for a.e. } \omega \text{ in } \Omega. \tag{10}$$

Also, thanks to (3), we have, by a standard formula for transformations,

$$\int_Y \Phi(y)\delta(\omega)(dy) = \int_{\mathbb{R}^{m+1}} \xi \; \delta^\Phi(\omega)(d\xi) \tag{11}$$

for a.e. ω in Ω. Here (3) implies

$$\int_\Omega [\int_{\mathbb{R}^{m+1}} \sum_{i=1}^{m+1} |\xi^i| \delta^\Phi(\omega)(d\xi)]\mu(d\omega) \leq \theta \; (\sum_{i=1}^{m} \|x_i\| + 1) < +\infty.$$

Define $g_{\Gamma_0} : \Omega^{na} \times \mathbb{R}^{m+1} \to [0, +\infty]$ by

$$g_{\Gamma_0}(\omega, \xi) = \begin{cases} |\xi| & \xi \in \Gamma_0(\omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where the Euclidean norm is used on $\mathbb{R}^{m+1}$. In combination with (10), the previous inequality implies

$$\int_{\Omega^{na}} [\int_{\mathbb{R}^{m+1}} g_{\Gamma_0}(\omega, \xi)\delta^\Phi(\omega)(d\xi)]\mu(d\omega) < +\infty.$$

By Lemma 4.6 this allows us to invoke Theorem 2.2 of [8], which has been recalled in the appendix as Theorem A.1. This gives

$$\int_{\Omega^{na}} [\int_{\mathbb{R}^{m+1}} \xi \; \delta^\Phi(\omega)(d\xi)]\mu(d\omega) \in \text{co} \int_{\Omega^{na}} \Gamma_0 \; d\mu = \int_{\Omega^{na}} \Gamma_0 \; d\mu.$$

Here Richter's theorem – i.e., essentially Lyapunov's theorem – guarantees the convexity of the set on the right. Since that set is an Aumann integral, its definition means that there must exist γ_0 in $\mathcal{L}^1_{\mathbb{R}^{m+1}}(\Omega^{na})$ such that $\gamma_0(\omega) \in \Gamma_0(\omega)$ for a.e. ω in Ω^{na} and

$$\int_{\Omega^{na}} [\int_{\mathbb{R}^{m+1}} \xi \; \delta^\Phi(\omega)(d\xi)]\mu(d\omega) = \int_{\Omega^{na}} \gamma_0 \; d\mu.$$

By the identity (11), this can also be written as

$$\int_{\Omega^{na}} [\int_Y \Phi(y)\delta(\omega)(dy)]\mu(d\omega) = \int_{\Omega^{na}} \gamma_0 \, d\mu. \tag{12}$$

In view of Lemma 4.5, we may invoke the implicit measurable function result of [15, Theorem III.38]. It follows that there exists a measurable function $f_H : \Omega^{na} \to Y$ such that the following hold for a.e. ω in Ω^{na}: (i) $f_H(\omega) \in F_0(\omega)$, (ii) $\gamma_0^i(\omega) = \langle x_i, f_H(\omega)\rangle$ for $i = 1, \ldots, n$ and (iii) $\gamma_0^{m+1}(\omega) \geq \|f_H(\omega)\|^*$. By (iii), (12) now gives

$$\theta \geq \int_{\Omega^{na}} [\int_Y \|y\|^* \delta(\omega)(dy)]\mu(d\omega) = \int_{\Omega^{na}} \gamma_0^{m+1} \, d\mu \geq \int_{\Omega^{na}} \|f_H\|^* \, d\mu,$$

which proves $f_H \in \mathcal{L}^1_Y(\Omega^{na})[X]$. Combining (i)-(ii) with (6) and (12) gives

$$\int_{\Omega^{na}} \langle x_i, f\rangle \, d\mu = \int_{\Omega^{na}} \langle x_i, f_H\rangle \, d\mu, \; i = 1, \ldots, m.$$

On the purely atomic part Ω^{pa} we set $f_H(\omega) := \hat{f}(\omega)$; then (8) shows that actually $f_H = f$ a.e. on Ω^{pa}. By definition of b, this gives the desired property of f_H. □

Following Lemma 4.4, we already concluded that (2) had been validated. We can now conclude that (1) has been proven as well: write $a = b + (a - b)$ and recall (9). Thus, the proof of Corollary 4.1 has been completed.

Next, we prove Theorem 3.1 by means of its own Corollary 4.1. The key is the following lemma, which is an obvious adaptation to the present context of a similar lemma of Hess [22, Lemmas 2.1, 4.1], which figures as Lemma 5.1 in [12].

Lemma 4.8. *The effective domain* dom $(s(\cdot \mid L))$ = dom $(s(\cdot \mid w^*\text{-cl co } L)) \subset X$ *has a nonempty interior for the norm topology, and for every x_0 in this interior there exists a constant γ_L such that for every w^*-compact convex $K \subset Y$, every $r \geq 0$ and every $y \in K + r\, L$*

$$\|y\|^* \leq \|K\|^* + \gamma_L[s(x_0 \mid K) + r - \langle x_0, y\rangle].$$

Proof. By [15, Corollary I.15], the interior of dom $(s(\cdot \mid L))$ = dom $(s(\cdot \mid w^*\text{-cl co } L))$ is nonempty. Fix an element x_0 in this interior; then, by the same result and the w^*-compactness of K, the set

$$W_\beta := \{y \in Y : y \in K + r\, L \text{ and } \langle -x_0, y\rangle \leq \beta\}$$

is w^*-compact for every $\beta \in \mathbb{R}$. If we choose β large enough, i.e., $\beta > \beta_0 := -s(x_0 \mid K + r\, L)$, then $W_\beta \neq \emptyset$ and there exists $\tilde{y} \in K + r\, L$ such that $\langle -x_0, \tilde{y}\rangle < \beta$. By this Slater-type condition, a well-known duality result [27] gives

$$s(x \mid W_\beta) = \min_{\lambda \geq 0} [s(x + \lambda x_0 \mid K + r\, L) + \lambda\beta].$$

for every $x \in X$. In particular, this implies $s(x \mid W_\beta) \leq s(x + x_0 \mid K + r\, L) + \beta$ for every $x \in X$ and $\beta > \beta_0$, so clearly

$$s(x \mid W_\beta) \leq s(x \mid K) + s(x_0 \mid K) + rs(x + x_0 \mid L) + \beta.$$

Since $s(\cdot + x_0 \mid L)$ is norm-continuous at the origin, there exists $\alpha_L > 0$ such that $s(x + x_0 \mid L) \leq 1$ for every $x \in X$ with $\|x\| \leq \alpha_L$. We conclude that

$$\alpha_L \|W_\beta\|^* = \sup_{x \in X, \|x\| \leq \alpha_L} s(x \mid W_\beta) \leq \alpha_L \|K\|^* + s(x_0 \mid K) + r + \beta$$

for $\beta > \beta_0$. Now let $y \in K + r\, L$ be arbitrary; then $\beta_0 \leq -\langle x_0, y\rangle$. Let $\beta_n := -\langle x_0, y\rangle + n^{-1}$; then $y \in W_{\beta_n}$ for every $n \in \mathbb{N}$, so the above gives $\alpha_L \|y\|^* \leq \alpha_L \|K\|^* + s(x_0 \mid K) + r + \beta_n$. In the limit the desired inequality is obtained, by setting $\gamma_L := \alpha_L^{-1}$. □

We can now start the proof of Theorem 3.1. Let $a \in w^*\text{-Ls}_k \int_\Omega F_k \, d\mu$ be fixed and arbitrary. By definition of the set $w^*\text{-Ls}_k \int_\Omega F_k \, d\mu$ there exists a subsequence $(F_{k_j})_j$ of (F_k) and an associated sequence $(f_{k_j})_j$ in $\mathcal{L}^1(\Omega; Y)[X]$ such that $a = w^*\text{-}\lim_j \int_\Omega f_{k_j} \, d\mu$ and for every $j \in \mathbb{N}$ one has $f_{k_j}(\omega) \in F_{k_j}(\omega) \subset G_{k_j}(\omega) + r_{k_j}(\omega) L$ for a.e. ω in Ω. By Lemma 4.8 one gets

$$\int_\Omega \|f_{k_j}\|^* d\mu \leq (1 + \gamma_L \|x_0\|) \int_\Omega^* \|G_{k_j}\|^* d\mu + \gamma_L \left(\int_\Omega r_{k_j} \, d\mu - \int_\Omega \langle x_0, f_{k_j}\rangle \, d\mu \right).$$

On the right, the sequences $(\int_\Omega^* \|G_{k_j}\|^* d\mu)_j$ and $(\int_\Omega r_{k_j} \, d\mu)_j$ are bounded because of (A_2) and uniform integrability of (r_k) in (A_1). Also, $(\int_\Omega \langle x_0, f_{k_j}\rangle d\mu)_j$ is bounded, since it converges to $\langle x_0, a\rangle$. This demonstrates that the sequence $(f_{k_j})_j$ meets Assumption (A') of Corollary 4.1. Therefore, application of that corollary gives

$$a \in \bigcap_{H \in \mathcal{H}} \left(\int_\Omega F_0 \, d\mu - C_0^* + H^\perp \right) \text{ and } a \in \int_{\Omega^{pa}} F_0 \, d\mu + \int_{\Omega^{na}} w^*\text{-cl co } F_0 \, d\mu - C_0^*.$$

We claim that $\text{dom }(s(\cdot \mid L)) \cap -C_{00} \subset -C_0$. Indeed, for every $x \in \text{dom }(s(\cdot \mid L)) \cap -C_{00}$

$$\langle -x, f_{k_j}(\omega)\rangle \geq -\max\{0, s(x \mid G_{k_j}(\omega))\} - r_{k_j}(\omega) s(x \mid L)$$

holds for a.e. ω in Ω. Because of the definition of C_{00} and the uniform integrability of $(r_k)_k$, this shows that $(\max\{0, \langle -x, f_{k_j}\rangle\})_j$ is uniformly integrable. So $x \in -C_0$, which proves our claim. From this we obtain $-C_0^* \subset \text{As}(\text{cl}(L - C_{00}^*)) = \text{As}(L - C_{00}^*)$ by taking polars; also, $F_0 \subset F_{00}$ a.e. is obvious. In view of what was reached above, this finishes the proof of Theorem 3.1.

A. An extension of Lyapunov's theorem

We recall the following theorem from [8] for Young measures in $\mathcal{Y}(\Omega; \mathbb{R}^d)$, where $d \in \mathbb{N}$. As demonstrated in Propositions 3.1 and 3.2 of [8], this result generalizes Aumann's identity for integrals of multifunctions and also yields an extension of Lyapunov's theorem.

Theorem A.1 ([8, part one of Theorem 2.2]). *Let $\Gamma : \Omega \to 2^{\mathbb{R}^d}$ be a multifunction with measurable graph and closed values. Define $g_\Gamma : \Omega \times \mathbb{R}^d \to [0, +\infty]$ by*

$$g_\Gamma(\omega, \xi) = \begin{cases} |\xi| & \text{if } \xi \in \Gamma(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Here the Euclidean norm is used on $\mathbb{R}^d$. Let $\eta \in \mathcal{Y}(\Omega; \mathbb{R}^d)$ satisfy

$$\int_\Omega \left[\int_{\mathbb{R}^d} g_\Gamma(\omega, \xi) \eta(\omega)(d\xi) \right] \mu(d\omega) < +\infty.$$

Then the barycenter bar $\eta(\omega) := \int_{\mathbb{R}^d} \xi\, \eta(\omega)(d\xi)$ *exists for a.e.* ω *in* Ω *and*

$$\int_\Omega \text{bar } \eta(\omega)\mu(d\omega) \in \text{co} \int_\Omega \Gamma\, d\mu.$$

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