Journal of Concrete and Applicable Mathematics
Vol. 3, N. 4, 481-499 (2005)

A note on a Liapounov-like theorem for some finitely additive measures and applications

Anna Martellotti – Anna Rita Sambucini

Dipartimento di Matematica e Informatica
1, Via Vanvitelli - 06123-I, Perugia (ITALY)
e-mail: amart@dipmat.unipg.it, matears1@unipg.it

Abstract

We give some results about the convexity of a pair of finitely additive measures and we apply them to derive the convexity of the Aumann integral of a suitable multifunction.

1991 AMS Mathematics Subject Classification: **28B20, 26E25, 46B20, 54C60**
Key words: Aumann integral, finitely additive measures, selections, convex measures, Stone isomorphism.

1. Introduction

The classical core-Walras equivalence result [13, Theorem II-2.1] is stated for a finite-dimensional commodity space, and a space of agents (Ω, Σ, μ) represented by a non-atomic, positive, countably additive measure space. This celebrated result of Equilibrium Theory has been extended in several directions: in particular ([1, 3, 28]) discuss and face the problem of extending it to the case of *strongly non-atomic* finitely additive measures. The proof of the classical result is based upon an idea of Aumann, and makes use of the geometrical and topological properties of the multivalued integral of a suitable

multifunction ranging on the commodity space.
The Aumann integral of a Banach-valued multifunction with respect to a finitely additive measure has been considered in [21] and [22]: in [21] the investigation concerned integrands with compact and convex values, while [22] examined the case of integrands with weakly compact and convex values; (see also [26] for a survey on this topic).

Recently, in [23] we have examined the case of multifunctions of the type

$$F(\omega) = [\Gamma(\omega) - e(\omega)] \cup \{0\}, \tag{1}$$

where Γ is a simple multifunction with values in the hyperspace of closed and convex values of a Banach space X, $e \in L^1_\mu(X)$ and μ is non atomic and countably additive.

Integrands of this form are those that occur in the core-Walras equivalence: the assumption on Γ has a meaningful interpretation from the point of view of the economic model.

The properties of the multivalued integral are usually achieved applying the classical Liapounov Theorem, together with its infinite dimensional versions. It is well known that Liapounov Theorem does not extend to finitely additive measures: weakened forms of it in this more general setting have been obtained by several authors [18, 6, 7, 29, 8, 2, 3]. A survey of the most important results can be found in [19].
What we obtained in [23] was that, if e has Liapounov indefinite integral, then F has convex Aumann integral. The proof of the result was based upon the following result
Theorem A ([23] Theorem 3.7) *Let X and Y be two Banach spaces, with X satisfying the* (RNP), *$\mu : \Sigma \to \mathbb{R}^+_0$ a non-atomic countably additive measure, $f = \sum_{i=1}^{n} x_i 1_{E_i}$ a Y-valued, simple function, and $n_2 = \int_{\cdot} e d\mu$ an X-valued Liapounov measure. Then, setting $n_1 = \int_{\cdot} f d\mu$, the range of the pair (n_1, n_2) is convex and compact in $Y \times X_w$.*
This paper is concerned with the Aumann integral for integrands of the form (1), but when μ is simply finitely additive: in this setting Theorem A above is in general false; a counterexample can be obtained by means of the results in [8].

However we shall show, through a completely different path, that the convexity of the Aumann integral of F can be reobtained also in the finitely additive setting, when the commodity space X is a Banach lattice and the indefinite integral of e satisfies suitable assumptions.

2. Preliminaries and definitions

Throughout this paper X will be a reflexive, separable Banach lattice, X^+ its positive cone. With X^* we denote the topological dual and with $X_1,\ X_1^*$ the unit balls of X and X^* respectively. We denote by X_w the space X equipped with its weak topology.
Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\mu : \Sigma \to [0, +\infty[$ a finitely additive bounded measure. In accordance to [11, Chapter III] we denote by $L^1_\mu(X)$ the space of X-valued, μ-integrable functions f. When $X = \mathbb{R}$ we shall simply write L^1_μ. Throught the paper we will use the symbol μ to denote a scalar measure, while with the symbol m we denote a vector valued one.

Definition 1 A finitely additive vector measure $m : \Sigma \to X$ is called *Liapounov* if, for every $E \in \Sigma$, $\quad m(\Sigma_E) := \{m(A), A \in \Sigma \cap E\}$ is convex and weakly compact for every $E \in \Sigma$. Since we have assumed that X is a reflexive Banach space it is enough to assume that $m(\Sigma_E)$ is bounded, closed and convex for every $E \in \Sigma$. If, for every $E \in \Sigma$, $m(\Sigma_E)$ is only convex, we will say that m is a *convex measure.* If, for every $E \in \Sigma$, there exists $B \in \Sigma_E$ such that $m(B) = \frac{1}{2}m(E)$, we will say that m is a *semiconvex measure.*

We remind that for a scalar finitely additive, bounded measure $\mu : \Sigma \to [0, +\infty)$ the conceps of strong continuity, semiconvexity and Liapounov are equivalent [8, 18]; where μ is *strongly continuous* if for every $\varepsilon > 0$ there exists a finite decomposition of Ω, $A_i \in \Sigma,\ i = 1, \dots, n$ such that $\mu(A_i) \leq \varepsilon$.

For a finitely additive vector measure we will denote by $|m|$ the variation of m, defined for every $E \in \Sigma$, by:

$$|m|(E) = \sup_{\Pi} \sum_{A_i \in \Pi} \|m(A_i)\|$$

where the supremum is taken over all the finite decompositions Π of the set E. Moreover we denote by $\|m\|$ its semivariation, given by

$$\|m\|(E) = \sup\{|x^*m|(E),\ x^* \in X^*,\ \|x^*\| \leq 1\}.$$

It is known that:

$$\sup\{\|m(A)\|, A \in \Sigma \cap E\} \leq \|m\|(E) \leq 4 \sup\{\|m(A)\|, A \in \Sigma \cap E\},$$

see for example [9, Proposition 1.1].

In the framework of [27] we will make use of the Stone extension; more precisely $\mathcal{G}$ will be the Stone algebra associated to Σ and $\tau : \Sigma \to \mathcal{G}$ the Stone isomorphism.
With $\overline{\mu} : \mathcal{G}_\sigma \to [0, +\infty[$ we will denote the extended measure of μ, where $\mathcal{G}_\sigma$ is the σ-algebra generated by $\mathcal{G}$. Observe that if μ is strongly continuous then its Stone extension $\overline{\mu}$ is non atomic and therefore Liapounov. Moreover, by [12], if $f \in L^1_\mu(X)$ then it is possible to define its Stone extension as a map $\overline{f} \in L^1_{\overline{\mu}}(X)$ such that for every $E \in \Sigma$,

$$\int_E f d\mu = \int_{\tau(E)} \overline{f} d\overline{\mu} \tag{2}$$

where the left hand side is defined in accordance to [11, Chapter III].
As a consequence if $f \in L^1_\mu(X)$ has Liapounov indefinite integral, its Stone extension $\overline{f}$ has the same property. It was also showed in [15] that $\|\overline{f}\| = \overline{\|f\|}$ $\overline{\mu}$-almost everywhere.

Let $m : \Sigma \to X$ be a vector-valued finitely additive measure. We say that m is *s-bounded* if $\lim_{n\to\infty} m(A_n) = 0$ for every sequence $(A_n)_n$ of pairwise disjoint sets in Σ.

Definition 2 A positive finitely additive measure $\sigma : \Sigma \to [0, \infty[$ is a *control* for m if and only if $\|m\| \sim \sigma$, in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following implications hold:

- if $\sigma(A) < \delta$ then $\|m\|(A) < \varepsilon$;
- if $\|m\|(A) < \delta$ then $\sigma(A) < \varepsilon$.

A control σ is said to be a *Rybakov control* if there exists a functional $x^* \in X^*$ such that $\sigma = |x^* m|$.

Remark 1 In [10, 25] the following equivalences were proved: a finitely additive measure m is s-bounded if and only if there exists a control for m if and only if there exists a Rybakov control for m. If m is also of bounded variation then its variation is equivalent to a Rybakov control for m. The following proof of this equivalence was communicated to us by one of the referees. If $\sigma = |x^* m|$ then obvioulsy $\sigma \ll |m|$. To prove the converse, for every $F = \{x_1^*, \dots, x_n^*\}$ in X_1^* let $\eta_F := \bigvee_{i=1}^n |x_i^* m|$ be the lattice supremum of the finitely additive measures $|x_1^* m|, \dots, |x_n^* m|$. Since all the η_F are dominated by $|m|$, then they are uniformly s-bounded and therefore they are uniformly σ-continuous. Hence the set-wise supremum $|m|(E) = \sup\{\eta_F(E), F \subset X_1^*, F \text{ finite } \}$ is σ-continuous. So, in this case, $\|m\| \sim |m|$. Moreover if m and σ are countably additive then the $\varepsilon - \delta$ absolute continuity is equivalent to $0 - 0$ absolute continuity.

3. A Liapounov result

We will now show that if m is X^+-valued and s-bounded then it admits a Rybakov control of the form $\sigma = y^* m$ for some $y^* \in (X^*)^+$.

Lemma 1 *If $m : \Sigma \to X^+$ is a s-bounded finitely additive measure then there exists $y^* \in (X^*)^+$ such that $\|m\| \sim y^* m$.*

Proof: Let $\sigma = |x_0^* m|$ be a Rybakov control for m, and let $y^* = |x_0^*|$ in the Banach lattice X^*.

Then, easily, $|x_0^*(x)| \le y^*(x)$ for every $x \in X^+$. Let now $\varepsilon > 0$ be fixed and consider δ according to the absolute continuity of $\|m\|$ with respect to σ; let $A \in \Sigma$ be such that $y^* m(A) \le \delta$; we want to show that $\|m\|(A) \le \varepsilon$. Indeed let Π be an arbitrary finite partition of A, since m is X^+-valued we have

$$\sum_{B \in \Pi} |x_0^* m(B)| \le \sum_{B \in \Pi} y^* m(B) = y^* m(A) \le \delta.$$

Taking the supremum with respect to Π we have $\sigma(A) \le \delta$, which in turn yields $\|m\|(A) \le \varepsilon$.

Conversely we prove now that $y^* m \ll \|m\|$. Let $\varepsilon > 0$ be fixed and consider

$\delta = \varepsilon \|y^*\|^{-1}$. If $\|m\|(A) < \delta$ then $y^* m(A) \leq \|y^*\| \, \|m\|(A) \leq \|y^*\| \, \delta < \varepsilon$. $\square$

We will show now that, also in the finitely additive case, if m is of bounded variation it is possible to find a control which is equivalent to the variation of m.

Proposition 1 *If $m : \Sigma \to X^+$ is a finitely additive measure with bounded variation, then there exists $y^* \in (X^*)^+$ such that $|m| \sim y^* m$.*

Proof: By Lemma 1 there exists $y^* \in (X^*)^+$ such that $y^* m$ is a Rybakov control, namely $y^* m \sim \|m\|$. Since m is of bounded variation then, by Remark 1, $\|m\|$ is equivalent to $|m|$. This concludes the proof. $\square$

Proposition 2 *Let $m : \Sigma \to X^+$ be an s-bounded finitely additive measure. The following are equivalent:*

2.1 *m is semiconvex;*

2.2 *m admits a filtering family, namely for every $B \in \Sigma$ there exists a filtering family $\{B_t\}_{t \in [0,1]}$ such that*

- **a)** *$B_0 = \emptyset$, $B_1 = B$ and, if $t < t'$, then $B_t \subset B_{t'}$;*
- **b)** *$m(B_t) = tm(B)$, for every $t \in [0, 1]$;*

2.3 *m is a convex measure.*

Proof: 2.1) $\Longrightarrow$ 2.2). Let $B \in \Sigma$ be fixed. With a standard argument it is possible to construct a filtering sequence $(B_t)_t, t \in \mathbb{Q}(2)$ which satisfies conditions a) and b), see for example [6, Lemma 2.1]. Let now $t \in]0, 1[$ be fixed, with $t \notin \mathbb{Q}(2)$, and let $(p_n)_n, (q_n)_n$ be two sequences in $\mathbb{Q}(2)$ such that $p_n \uparrow t$ and $q_n \downarrow t$. Put $B'_t = \cup_n B_{p_n}$, $B''_t = \cap_n B_{q_n}$ and note that $B'_t \subseteq B''_t$; hence

$$m(B'_t) \geq \sup_n m(B_{p_n}) = tm(B) = \inf_n m(B_{q_n}) \geq m(B''_t) \geq m(B'_t).$$

Therefore we can choose for instance $B_t = \cap_n B_{q_n}$.

2.2) $\Longrightarrow$ 2.3). Let $A, B \in \Sigma$ be fixed and let $t \in [0, 1]$. As in [7, Theorem 2.4] let

$$C_t = (B \setminus A)_t \cup (A \cap B) \cup (A \setminus B)_{1-t},$$

where the families $\{(B \setminus A)_t\}_t, \{(A \setminus B)_t\}_t$ are the filtering families for $B \setminus A$ and $A \setminus B$ respectively. We have that $C_0 = A, C_1 = B$ and

$$\begin{aligned} m(C_t) &= tm(B \setminus A) + m(A \cap B) + (1-t)m(B \setminus A) = \\ &= tm(B) + (1-t)m(A) \end{aligned}$$

2.3) $\Longrightarrow$ 2.1). It is obvious. □

The equivalence between 2.1) and 2.2) in the finite dimensional case had already been obtained in [7].

We shall suppose now that m is the indefinite integral of a function $e \in L^1_\mu(X^+)$ which satisfies the following assumption:

(h) μ is a *control* for m.

Remark 2 A sufficient condition for **(h)** is, for example, the following:

$$\operatorname*{ess\,inf}_{\Omega} \|e\| \geq r > 0 \tag{3}$$

The assumption (3) has a very meaningful interpretation in the application to the economic model namely, when e is the initial endowment. We could label it as *minimal entrance feee* (m.e.f.) because from the economic point of view this means that all the consumers, except a set of measure zero, have a minimal granted support r.

Observe also that m.e.f. assumption is stronger than the condition **(h)**. For instance, consider in the interval $[0,1]$, $\mu : \Sigma \to [0,1]$ defined as

$$\mu(A) = \int_A \frac{1}{2\sqrt{x}} dx,$$

where dx is the Lebesgue measure and A is a Lebesgue measurable set and take $e(x) = 2\sqrt{x}$.
The measure $m = \int e d\mu$ coincides with the Lebesgue measure and admits μ as a control, but e does not satisfy m.e.f. assumption.

We want now to prove the announced result:

Proposition 3 *Let μ be a s-bounded, non negative, finitely additive measure, and let $e \in L^1_\mu(X^+)$ be such that $m = \int e d\mu$ is semiconvex and μ is a control for m; then the pairs (m, μ) and $(m, x\mu)$ are convex for every $x \in X$.*

Proof: The idea of the proof is analogous to that given in [7, Propositions 2.4 and 2.5]. First of all we prove that the pair $(m, |m|)$ is semiconvex.
Let $E \in \Sigma$ be fixed and consider the filtering family $\{E_t\}_{t\in[0,1]}$ associated to m and E. If $|m|(E_{1/2}) = 2^{-1}|m|(E)$ then we have semiconvexity. Otherwise let, for example, $|m|(E_{1/2}) < 2^{-1}|m|(E)$ and therefore $|m|(E \setminus E_{1/2}) > 2^{-1}|m|(E)$. If we apply Proposition 2 to m we find the filtering families $\{A_t\}_t, \{B_t\}_t$ associated to $E_{1/2}$ and $E \setminus E_{1/2}$ respectively. We set

$$C_t = A_t \cup B_{1-t}.$$

By construction, for every t

$$m(C_t) = tm(E_{1/2}) + (1-t)m(E \setminus E_{1/2}) = \frac{1}{2}m(E).$$

We prove now that $|m|(C_t)$ is a continuous function in t. Let y^*m be a Rybakov control for m as in Proposition 1. Therefore, for $\varepsilon > 0$ fixed, let $\delta > 0$ be that of the absolute continuity of $|m|$ with respect to y^*m. Let $t, s \in [0,1]$ be such that $|t-s|\ y^*m(E) \leq \delta$; suppose for instance $t < s$. The set $C_t \,\Delta\, C_s$ is given by $(A_s \setminus A_t) \cup (B_{1-t} \setminus B_{1-s})$. Since $y^*m(C_t \,\Delta\, C_s) = |t-s|y^*m(E) < \delta$ we have that

$$|\ |m|(C_t) - |m|(C_s)\ | \quad \leq \quad |m|(C_t \,\Delta\, C_s) < \varepsilon.$$

Hence, by continuity, as $|m|(C_0) < 2^{-1}|m|(E)$, while $|m|(C_1) > 2^{-1}|m|(E)$, there exists $t \in]0,1[$ such that $|m|(C_t) = 2^{-1}|m|(E)$.
Since μ is a control for m this implies also the continuity of $t \mapsto \mu(C_t)$ and then the semiconvexity of (m, μ). Finally the convexity of the pair (m, μ) follows by Proposition 2 if we consider $E = X \times I\!R$, $E^+ = X^+ \times [0, \infty[$, and the fact that the finitely additive measure (m, μ) is s-bounded. The convexity of $(m, x\mu)$ is an immediate consequence of the definition and of the convexity of (m, μ).

4. Applications to multivalued finitely additive integral of non convex integrands

Throughout this section, and similarly as in [23], we will adopt the following notations:

(i) (Ω, Σ) is a measurable space and $\mu : \Sigma \to [0, \infty[$ a s-bounded finitely additive measure which is also strongly continuous.

(ii) $(cf(X), h)$ and $(cwk(X), h)$ are the families of non empty, convex, closed (non empty, convex and weakly compact respectively) subsets of X with the Hausdorff distance.

(iii) $\Gamma = \sum_{i=1}^{p} C_i 1_{E_i}$ is a simple multifunction with closed and convex values with $E_i \cap E_j = \emptyset$ for $i \neq j$;

(iv) $e \in L^1_\mu(X^+)$ is such that $\lambda(E) := \displaystyle\int_E e d\mu$ is a convex finitely additive measure;

(v) $G = (\Gamma - e),\ F = G \cup \{0\}$.

We examine now the problem studied in [23] when the measure with respect to which we integrate is only finitely additive. This case is not a mere extension of the countably additive one. It has applications for instance in finitely additive economies which were introduced first in [1] by Armstrong and Richter: they explained why their model of a large economy is more realistic than the countably additive one. The same model was also extensively studied in [3], [4].

We begin with the bounded case, namely we assume that $C_i \in cwk(X)$ for every $i = 1, \dots, p$. Thanks to the Radström Embedding Theorem ([24]) and to the Stone isomorphism we can consider the multifunctions $\overline{\Gamma}$ and $\overline{G}$ where

$$\overline{\Gamma} = \sum_{i=1}^{p} C_i 1_{h(E_i)}, \tag{4}$$

$$\overline{G} = \overline{\Gamma - e} = \sum_{i=1}^{p} C_i 1_{h(E_i)} - \overline{e}. \tag{5}$$

Thanks to (v) above, F certainly admits totally measurable selections and so we can define the Aumann integral as usual, using the finitely additive μ-integrability, [11, Chapter 3]: namely

- S^1_F is the set of all μ-integrable selections of a multifunction F, that is

$$S^1_F = \{f \in L^1_\mu(X) : f(\omega) \in F(\omega) \quad \mu - \text{almost everywhere}\};$$

- the Aumann integral of F is defined by

$$(A) - \int_E F d\mu = \left\{ \int_E f d\mu,\ f \in S^1_F \right\}.$$

The finitely additive case here considered is quite different from the case considered, for instance, in [21] and [22], where the existence of suitable selections depends upon the topological properties of the values of F.

Moreover, we shall denote by $M_\Gamma,\ M_G$ the finitely additive multimeasures defined as the indefinite Aumann integrals of Γ and G respectively.
By $R_X(M_{(\cdot)})$ we shall denote the range of $M_{(\cdot)}$ that is

$$R_X(M_{(\cdot)}) = \bigcup_{E \in \Sigma} M_{(\cdot)}(E).$$

As Γ takes values in $cwk(X)$, its Aumann integral is convex and weakly compact: in fact, for every $E \in \Sigma$, we have that

$$\begin{aligned} \sum_{i=1}^{p} C_i \mu(E \cap E_i) \quad &\subseteq \quad (A) - \int_E \Gamma d\mu \subseteq (A) - \int_{h(E)} \overline{\Gamma} d\overline{\mu} = \\ &= \quad \sum_{i=1}^{p} C_i \overline{\mu}(h(E \cap E_i)), \end{aligned} \tag{6}$$

where the first inclusion and the last equality can be obtained by [9, Corollary 8] since in the proof of this corollary the countable additivity is not required, while the middle inclusion can be obtained analogously to the proof of [21, Theorem 5.1] (note that, since Γ is simple, we do not need the compactness of the values of Γ).

Using these facts, statements analogous to those of [23, Propositions 3.1, 3.2 and 3.4], with the strong continuity of μ replacing the non atomicity, hold for Γ also in the finitely additive case. Namely

(a) $M_\Gamma(E) = \left\{ \int_E s d\mu, \; s \text{ is a simple selection of } \Gamma \right\};$

(b) $R_X(M_\Gamma) \in cwk(X);$

(c) $M_G(E) = (A) - \int_E \Gamma d\mu - \int_E e d\mu \in cwk(X)$ for every $E \in \Sigma$.

In [23], Theorem 3.7 allowed us to obtain that $R_X(M_G) \in cwk(X)$ when μ is non atomic and countably additive. In the finitely additive case, Theorem 3.7 which is, in fact, a Lyapounov-type statement, does not hold in its complete extension (a counterexample is easily derived from [8, Theorem 4.4]). Its weakened finitely additive version, that is Theorem 1 below, will play a similar role in achieving the convexity of $R_X(M_G)$ in our case, although weak compactness is not assured in general. These two results, despite their similarity, do not compare: here we have only finite additivity but we have to assume the equivalence between μ and m; the proof of Theorem 1 completely differs from that of the quoted countably additive version, and is heavily based upon the results of Section 3.

Theorem 1 *Let X be a Banach lattice, Y a Banach space, μ a strongly continuous finitely additive measure and $e \in L^1_\mu(X^+)$ be such that the finitely additive measure $\lambda = \int e d\mu$ is convex and admits μ as a control. If f is a Y-valued simple function then setting $m = \int f d\mu$, the range of the pair (m, λ) is convex in $Y \times X$.*

Proof: We know that if $f = \sum_{i=1}^{p} c_i 1_{E_i}$ for some finite decomposition of Ω, $\{E_1, \ldots, E_p\}$ then

$$(m, \lambda)(\Sigma_E) = (m, \lambda)(\Sigma_{E \cap E_1}) + \ldots + (m, \lambda)(\Sigma_{E \cap E_p}).$$

So it is enough to note that, from Proposition 3, for each $i = 1, \ldots p$, $(m, \lambda)(\Sigma_{E \cap E_i})$ is convex and m is a multiple of μ on $\Sigma_{E \cap E_i}$. □

As an immediate consequence we obtain that

Corollary 1 *Let $f_j = \sum_{i=1}^{p} z^i_j 1_{E_i}, j = 1, 2$ be simple and measurable functions with values in X and let λ as in Theorem 1. Then setting $m_j = \int f_j d\mu - \lambda$ we have that $m_1, m_2, (m_1, m_2)$ are convex finitely additive measures.*

Proof: Applying Theorem 1, one easily deduces that m_1 and m_2 are convex finitely additive measures. We shall prove that the finitely additive measure (m_1, m_2) is convex. It is enough to prove that from Proposition 3, for each $i = 1, \dots p$, $(m_1, m_2)(\Sigma_{E\cap E_i})$ is convex for every $E \in \Sigma$.
By Proposition 3 the pair (λ, μ) is convex. So, for every $D_1, D_2 \in \Sigma_{E\cap E_i}$ and for every $t \in]0,1[$ there exists $C \in \Sigma_{E\cap E_i}$ such that

$$\begin{aligned} \mu(C) &= t\mu(D_1) + (1-t)\mu(D_2) \\ \lambda(C) &= t\lambda(D_1) + (1-t)\lambda(D_2). \end{aligned}$$

Since on $\Sigma_{E\cap E_i}$ $m_j = z_j^i \mu$ we have, for $j = 1, 2$,

$$\begin{aligned} m_j^i(C) &= z_j^i \mu(C) - \lambda(C) = \\ &= tz_j^i \mu(D_1) + (1-t) z_j^i \mu(D_2) - t\lambda(D_1) - (1-t)\lambda(D_2) = \\ &= tm_j(D_1) + (1-t)m_j(D_2); \end{aligned}$$

thus

$$(m_1, m_2)(C) = t(m_1, m_2)(D_1) + (1-t)(m_1, m_2)(D_2). \qquad \Box$$

Because of Theorem 1 the set $R_X(M_G)$ is convex. In fact

Theorem 2 *Under the previous assumptions $R_X(M_G)$ is convex.*

Proof: It is possible to prove it in an analogous way as in [16, Lemma 7]. We report the proof for completeness. We recall that $G = \Gamma - e$, with $\Gamma = \sum_{i=1}^p C_i 1_{E_i}$ and $\lambda = \int e d\mu$.
Let $\varphi = \sum_{i=1}^p z_i 1_{E_i}$ be a fixed simple selection of Γ. Consider the finitely additive selection measure ν of M_G, defined as

$$\nu(\cdot) = \int_{\cdot} \varphi d\mu - \lambda.$$

Fix x_1, x_2 in $R_X(M_G)$ and $t \in]0,1[$. Then there exist two sets $A_1, A_2 \in \Sigma$ and two simple selections of Γ, f_1 and f_2 ($f_j = \sum_{i=1}^p z_j^i 1_{E_i}, j = 1, 2$), such that

$$x_j = \int_{A_j} f_j d\mu - \lambda(A_j), \qquad j = 1, 2.$$

We put $m_j = \int f_j d\mu - \lambda$, for $j = 1, 2$. By Corollary 1 $m_1, m_2, (m_1, m_2)$ are convex. Then there exist B_1, B_2, B_3 such that $B_1 \subset A_1 \setminus A_2$, $B_2 \subset A_2 \setminus A_1$ and $B_3 \subset A_1 \cap A_2$ and

$$\begin{aligned}
&m_1(B_1) = tm_1(A_1 \setminus A_2); \qquad m_2(B_2) = (1-t)m_2(A_2 \setminus A_1);\\
&(m_1, m_2)(B_3) = t(m_1, m_2)(A_1 \cap A_2).
\end{aligned}$$

Set $B_4 = (A_1 \cap A_2) \setminus B_3$ and $B = \cup_{i=1}^4 B_i$. We will show that for a suitable selection ν^* of M_G, $\nu^*(B) = tx_1 + (1-t)x_2$.
Indeed, for every $E \in \Sigma$, define

$$\begin{aligned}
\nu^*(E) &= \nu(E \setminus B) + m_1(E \cap (B_1 \cup B_3)) + m_2(E \cap (B_2 \cup B_4)) = \\
&= \int_{E \setminus B} \varphi d\mu - \lambda(E \setminus B) + \int_{E \cap (B_1 \cup B_3)} f_1 d\mu + \\
&+ \int_{E \cap (B_2 \cup B_4)} f_2 d\mu - \lambda(E \cap B) = \\
&= \int_E \left(\varphi \cdot 1_{E \setminus B} + f_1 \cdot 1_{E \cap (B_1 \cup B_3)} + f_2 \cdot 1_{E \cap (B_2 \cup B_4)}\right) d\mu - \lambda(E).
\end{aligned}$$

Then the finitely additive measure ν^* is a selection of M_G and if we evaluate ν^* on the set B we obtain

$$\begin{aligned}
\nu^*(B) &= m_1(B_1 \cup B_3) + m_2(B_2 \cup B_4) = tm_1(A_1 \setminus A_2) + tm_1(A_1 \cap A_2) + \\
&+ (1-t)m_2(A_2 \setminus A_1) + m_2(A_1 \cap A_2 \setminus B_3) = \\
&= tx_1 + (1-t)x_2.
\end{aligned}$$

□

Moreover we have that

Theorem 3 *If G is a multifunction as before then* $cl\{R_X(M_G)\} = R_X(M_{\overline{G}})$.

Proof: From (c), (5), (6) above and [22, Theorem 5.1], (which holds in our case without compactness of values of G), for every $E \in \Sigma, M_G(E) \subset M_{\overline{G}}(\tau(E))$.
For the converse inclusion, if $x \in R_X(M_{\overline{G}})$ then there exists a set $H \in \mathcal{G}_\sigma$ such that

$$x \in M_{\overline{G}}(H) = (A) - \int_H \overline{\Gamma} d\overline{\mu} - \overline{\lambda}\ (H)$$

and then $x = \sum_{i=1}^{p} x_i \overline{\mu}(H \cap \tau(E_i)) - \overline{\lambda}\ (H)$ for some $x_i \in C_i,\ i = 1, \dots, p$.
Since $\mathcal{G}$ is Fréchet-Nikodym dense in $\mathcal{G}_\sigma$, for every $\varepsilon > 0$ there exists a $\mathcal{G}$ -measurable set B such that

$$\overline{\mu}(H \Delta B) \leq \frac{\varepsilon}{2kp},$$

for some $k > \max\{\|e\|_1, h(C_1, \{0\}), \dots, h(C_p, \{0\})\}$. Then

$$\|x - \sum_{i=1}^{p} x_i \overline{\mu}(B \cap \tau(E_i)) - \overline{\lambda}\ (B)\| \leq \sum_{i=1}^{p} \|x_i\| \overline{\mu}(H \Delta B) + |\overline{\lambda}\ |(H \Delta B) \leq \varepsilon.$$

Since B is a $\mathcal{G}$ -measurable set, there exists $E \in \Sigma$ such that $\tau(E) = B$; let us put

$$x_\varepsilon := \sum_{i=1}^{p} x_i \overline{\mu}(B \cap \tau(E_i)) - \overline{\lambda}\ (B) = \sum_{i=1}^{p} x_i \mu(E \cap E_i) - \lambda(E).$$

We have that $x_\varepsilon \in R_X(M_G)$ and $\|x - x_\varepsilon\| \leq \varepsilon$.
Hence $R_X(M_{\overline{G}}) \subset cl\{R_X(M_G)\}$ and again applying (c), the equality follows. □

Now, as in [23, Theorem 3.15], one can show that, if $0 \notin G(\omega)$ for all $\omega \in \Omega$, then, for every $E \in \Sigma$,

$$(A) - \int_E F d\mu = R_X(M_G|_{\Sigma \cap E}) \tag{7}$$

and therefore it is convex.

Let us now consider the unbounded case, namely assume that $C_i \in cf(X),\ i = 1, \dots p$. For each integer n, consider

$$\Gamma_n(\omega) = \Gamma(\omega) \cap nX_1, \qquad F_n(\omega) = (\Gamma_n(\omega) - e(\omega)) \cup \{0\}.$$

As in [23, Proposition 3.16] one shows that, for every $E \in \Sigma$

$$(A) - \int_E F d\mu = \bigcup_n\ (A) - \int_E F_n d\mu. \tag{8}$$

The sequence on the right hand side of (8) is increasing. Hence, if $0 \notin G$, $(A) - \int_E F d\mu$ is the union of an increasing sequence of convex sets, and therefore is convex.

For the general case, that is when some values of G contain 0, similarly to [23, Theorem 3.18] denote by Ω_0 the set $\{\omega : 0 \in G(\omega)\}$; the Aumann integral splits into two parts:

$$(A)-\int_E F d\mu = (A)-\int_{E\cap\Omega_0} F d\mu + (A)-\int_{E\setminus\Omega_0} F d\mu$$

It only remains to derive the convexity of the first summand by noting that F has convex values in Ω_0.

In conclusion we have obtained the following:

Theorem 4 *Let (Ω, Σ, μ) be a non negative finitely additive measure space with μ strongly continuous, X a Banach lattice, $\Gamma : \Omega \to cf(X)$ a simple multifunction and $e \in L^1_\mu(X^+)$ generate a semiconvex finitely additive measure which admits μ as a control. Consider $F = (\Gamma - e) \cup \{0\}$. Then, for every $E \in \Sigma$, $(A)-\int_E F d\mu$ is convex.*

We shall now derive a result similar to [23, Theorem 3.18]. Indeed, by means of Theorem 3, and the results in the countably additive case, [23, Theorem 3.18], we obtain the following

Theorem 5 *Let (Ω, Σ, μ) be a non negative finitely additive measure space with μ strongly continuous, X a Banach lattice, $\Gamma : \Omega \to cf(X)$ a simple multifunction and let $e \in L^1_\mu(X^+)$ generate a Liapounov indefinite integral which admits μ as a control. Consider $F = (\Gamma - e) \cup \{0\}$. Then, for every $E \in \Sigma$, $(A)-\int_E F d\mu$ is a convex set which is the union of an increasing sequence of convex and relatively weakly compact sets.*

Proof: The line of the proof is somewhat analogous to that of [23, Theorem 3.18]. As before consider

$$\Gamma_n(\omega) = \Gamma(\omega) \cap nX_1, \quad G_n(\omega) = \Gamma_n(\omega) - e(\omega), \quad F_n = G_n(\omega) \cup \{0\}.$$

and the Stone transforms of Γ_n, G_n, denoted by $\overline{\Gamma}_n$ and $\overline{G}_n$ respectively. From [23, Theorem 3.14] $R_X(M_{\overline{G}_n})$ is weakly compact and, since from (7),

$$(A)-\int_E F_n d\mu = R_X(M_{G_n|_{E\cap\Sigma}})$$

by Theorem 3 above, we have that $cl\left\{\int_E F_n d\mu\right\}$ is convex and weakly compact for each n. Hence the already proved equality

$$(A) - \int_E F d\mu = \bigcup_n (A) - \int_E F_n d\mu$$

shows the assertion. □

Remark 3 Note that Theorem 5 above is the only result that can be partially derived from the countably additive case [23] by means of the Stone extension. The other results in this section, despite the similarity of the statements, cannot be obtained in this way since the assumptions here and in [23] do not compare. For example if we take μ the Lebesgue measure on $[0,1]$, a non negative integrable function e such that $\mu(\text{supp } e) \in]0,1[$ then $\lambda = \int e d\mu$ is automatically Liapounov but $\mu \not\ll \lambda$. Viceversa the example which can be derived from [8] verifies the hypothesis on μ and λ given here but λ is not Liapounov.

Acknowledgment

This work was partially supported by the G.N.A.M.P.A. of I.N.D.A.M.
The authors would like to thank one of the referees for his suggestions and improvements of some of the results obtained in this paper.

References

[1] T.E. ARMSTRONG–M.K. RICHTER, The Core-Walras equivalence, *J. Econom. Theory,* **33**, 116-151, (1984).

[2] A. AVALLONE–A. BASILE, On the Liapunov–Richter theorem in finitely additive setting, *J. of Mathematical Economics,* **22**, 557-561, (1993).

[3] A. BASILE, Finitely additive correspondences, *Proc. of the Amer. Math. Soc.,* **121**, N. 3, 883-891, (1994).

[4] A. BASILE–M.G. GRAZIANO, Restricted coalition formation mechanism in finitely additive economies, preprint 2001.

[5] C. L. BYRNE, Remarks on the Set-Valued Integrals of Debreu and Aumann, *J. Math. Anal. Appl.,* **62**, 243-246, (1978).

[6] D. CANDELORO–A. MARTELLOTTI, Su alcuni problemi relativi a misure scalari subadditive e applicazioni al caso dell'additività finita, *Atti Sem. Mat. Fis. Univ. Modena,* **27**, 284-296, (1978).

[7] D. CANDELORO–A. MARTELLOTTI, Sul rango di una massa vettoriale, *Atti Sem. Mat. Fis. Univ. Modena,* **28**, 102-111,(1979).

[8] D. CANDELORO–A. MARTELLOTTI, Geometric Properties of the Range of Two-Dimensional Quasi-Measures with respect to Radon-Nikodym Property, *Adv. in Math.,***93** N.1, 9-24,(1992).

[9] J. DIESTEL–J.J. UHL, *Vector measures,* Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.

[10] L. DREWNOWSKI, On control submeasures and measures, *Studia Math.,* **50**, 203-224, (1974).

[11] N. DUNFORD–J. T. SCHWARTZ, *Linear Operators Part I,* Interscience, New York, 1958.

[12] C. FEFFERMANN, L_p spaces over finitely additive measures, *Pacific J. Math.,* **26**, 265-271, (1968).

[13] W. HILDEBRAND, *Core and Equilibria of a Large Economy,* Princeton University Press, Princeton, 1974.

[14] M. P. KATS, On the continuation of vector measures, *Sibirskii Mathematicheskii Zhurnal,* **13** N. 5, 1158-1168, (1972).

[15] M.C. ISIDORI–A. MARTELLOTTI–A.R. SAMBUCINI, Integration with respect to orthogonally scattered measures, *Math. Slovaca,* **48** N. 3, 253-269, (1998).

[16] LE VAN TU, The range of a multimeasure, *J. Australian Math. Soc.,* (Serie A) **26**, 442-452, (1978).

[17] J. LINDENSTRAUSS, A short proof of Liapounoff's convexity theorem, *J. of Mathematics and Mechanics,* **6** , 971–972, (1966).

[18] D. MAHARAM, Finitely additive measures on integers, *Indian J. of Statistic,* **38** Serie A, Pt.1, 44-59, (1976).

[19] A. MARTELLOTTI, Finitely additive Phenomena, *Rend. Ist. Mat. Univ. Trieste,* **33**, 201-249, (2001).

[20] A. MARTELLOTTI–K. MUSIAL–A.R. SAMBUCINI, A Radon-Nikodym theorem for the Bartle-Dunford-Schwartz Integral with respect to Finitely Additive Measures, *Atti Sem. Mat. Fis. Univ. Modena,* **42**, 625-633, (1994).

[21] A. MARTELLOTTI–A.R. SAMBUCINI, On the comparison between Aumann and Bochner integrals, *J. of Math. Anal. and Appl.,* **260**, 6-17, (2001).

[22] A. MARTELLOTTI–A.R. SAMBUCINI, The finitely additive integral of multifunction with closed and convex values, *Zeitschrift für Analysis ihre Anwendungen,* **21**, 851-864, (2002).

[23] A. MARTELLOTTI–A.R. SAMBUCINI, Multivalued integral of non convex integrands, *Internation J. of Pure and Applied Mathematics,* **5** N. 1, 11-28, (2003).

[24] H. RADSTRÖM, An Embedding Theorem for Spaces of Convex Set, *Proc. Amer. Math. Soc.* **3** , 165–169, (1952).

[25] V. RYBAKOV Theorem of Bartle, Dunford and Schwartz on vector valued measures, *Math. Notes,* **7***, 147-151, (1970).*

[26] A. R. SAMBUCINI, Remarks on set valued integrals of multifunctions with non empty, bounded, closed and convex values, *Comm. Math.,* **39**,153-165, (1999).

[27] R. SIKORSKI, *Boolean Algebras,* Springer Verlag, Berlin, 1964.

[28] K. VIND, Edgeworth Allocations in Exchange Economy with many Traders, *Int. Econom. Review,* **5** , 165-177, (1964).

[29] H. VOLKMER–H. WEBER, Der Wertebereich atomloser Inhalte, *Archive der Mathematik,* **40**, 464-474, (1983).