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The Perron Integral of order Two in Riesz Spaces via Peano Derivatives

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Abstract

A Perron-type integral of order two for Riesz-space-valued functions in terms of Peano derivatives is investigated. We give some fundamental properties of this integral and the integration by parts formula.

Key words: Riesz space, Perron integral, Peano derivative, major and minor function, integration by parts.

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1 Introduction

A Perron-type integral of higher order was introduced by R. D. James in [11, 12] to solve the problem of recovering the coefficients of convergent trigonometric series (see also [17]). A similar Perron integral was defined by P. S. Bullen in [9]. The difference between these two approaches is that James defined his integral in terms of generalized Riemann symmetric derivatives by major and minor functions satisfying "boundary-type" conditions while in [9] the Peano derivatives and some kind of an "initial" conditions were used instead.

Some results related to the theory of the Perron integral of order two in the context of Riesz spaces were given in [4, 6, 7], following the approach of [11], together with some applications to Fourier series and stochastic processes. In particular in [6] we obtained an integration by parts formula for this type of the Perron integral of order two. However attempts to extend this study of the Perron integral, defined in terms of the Riemann symmetric derivatives, to the case of higher orders meet some difficulties connected in particular with a problem whether (P^{k-1})-integrability implies (P^k)-integrability. This problem is still open for $k \geq 3$ in the case of Riesz-space-valued functions.

An alternative is the Bullen approach based on using the Peano derivatives instead of the Riemann ones. In the present paper we concentrate on the second order Perron integral based on the Peano derivatives. This permits us to obtain some results similar to those we have got in [6], under essentially weaker assumptions. Namely the hypotheses of boundedness of the derivatives of major and minor functions and super Dedekind completeness of the Riesz space, assumed in [6], can be dropped.

This paper is structured as follows: in Section 2 we introduce some preliminary notions, in particular we remind a concept of the global limit, very important for the whole theory. The second order Perron integral with respect to the Peano derivatives is introduced in Section 3. In Section 4 we compare this integral with some other integrals, in particular with the one considered in [6]. Finally in Section 5 we prove the integration by parts formula.

2 Preliminaries

For all the definitions related to Riesz spaces and their properties, the Maeda-Ogasawara-Vulikh representation theorem and the fundamental concepts of Differential Calculus we refer to [6, Sections 2,3,4]. We say that R is an *algebra* if it is a Dedekind complete Riesz space endowed with a structure of "product", satisfying distributive laws and compatibility with order. From now on, R is always an algebra, $[a, b]$ is a fixed compact interval of the real line $\mathbb{R}$ and E is a nonempty subset of $[a, b]$. We denote by Γ the set of all positive real-valued functions, defined on $[a, b]$. Furthermore, we say that a property is fulfilled *nearly everywhere* (shortly, n. e.) in $[a, b]$ if it is satisfied in the complement of a countable subset of $]a, b[$.

The concept of convexity is formulated as in the classical case (for references see for example [8]). We say that f is 0-*convex in* $[a, b]$ if $f(x) \geq 0$ for every $x \in [a, b]$; 1-*convex in* $[a, b]$, if and only if f is increasing in $[a, b]$; 2-*convex,* if it is convex.

We remind the concepts of (g)-limit (global limit), (g)-limsup and (g)-liminf (see [6, Definitions 3.1, 3.2]).

Let $\phi(x, h)$ ($x \in [a, b]$ and $h \in \mathbb{R} \setminus \{0\}$) be an R-valued function. We say that a *global limit*

$(g)\lim_{h\to 0}\phi(x,h)$ exists in E and is equal to $\widehat{\phi}(x)$ if

$$\inf_{\gamma\in\Gamma}\left[\sup\left\{\left|\phi(x,h)-\widehat{\phi}(x)\right| \,:\, x\in E, 0<|h|\le\gamma(x)\right\}\right]=0.$$

We say also that

$$(g)\limsup_{h\to 0}\ \phi(x,h)=\overline{\phi}(x) \qquad ((g)\liminf_{h\to 0}\ \phi(x,h)=\underline{\phi}(x))$$

is a *global limsup* (*global liminf*) in E if there is an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ (that is a net with $p_\gamma\downarrow 0$) such that for all $\gamma\in\Gamma$ and $x\in E$ we have:

$$0\le \sup\{\phi(x,h): 0<|h|\le\gamma(x)\}\ -\overline{\phi}(x)\le p_\gamma$$

$$(0\le \underline{\phi}(x)-\inf\{\phi(x,h): 0<|h|\le\gamma(x)\}\le p_\gamma).$$

Obviously $(g)\lim_{h\to 0}\phi(x,h)$, when it exists, is the common value of $(g)\limsup_{h\to 0}\ \phi(x,h)$ and $(g)\liminf_{h\to 0}\ \phi(x,h)$.

Now we define, for an R-valued function $\phi(x,h)$, the concept of $o(h^k)$. We say that $\phi(x,h)$ is *of the type* $o(h^k)$, $k\ge 0$, in E if there is an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that

$$\sup\{|\phi(x,h)| : x\in E,\ |h|\le\gamma(x)\}\le |h|^k\, p_\gamma \tag{1}$$

for all $\gamma\in\Gamma$ and $x\in E$, and we write $\phi(x,h)\in o^k(E)$ or $\phi(x,h)=o(h^k)$ in E.

Observe that (1) is equivalent to the condition

$$\inf_{\gamma\in\Gamma}\left[\sup\left\{\frac{|\phi(x,h)|}{|h|^k} : x\in E, 0<|h|\le\gamma(x)\right\}\right]=0, \tag{2}$$

that is in E

$$(g)\lim_{h\to 0}\frac{|\phi(x,h)|}{|h|^k}=0.$$

3 Peano derivatives and Perron integral

We apply the above notion of the global limit to define global derivatives.

A function $f:[a,b]\to R$ is (g)-*differentiable in* E if there exists a function $f':E\to R$ such that

$$\inf_{\gamma\in\Gamma}\left[\sup\left\{\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right| : x\in E, 0<|h|\le\gamma(x)\right\}\right]=0.$$

We use the following notation given for example in [8] for the real-valued case: $D^k f$ means the Riemann symmetric derivative, $D_{(k)}f$ the Peano derivative, while $f^{(k)}$ is the usual derivative. Namely, f' denotes in our case the usual global derivative of f and f'' the global derivative of f'.

In [6, Definition 6.1] we introduced the Riemann symmetric derivative of order 1 as the usual (g)-derivative and the one of order 2 as the global limit

$$D^2 f(x) := (g) \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

The *Peano derivative* of order 0 and 1 $D_{(0)}f$ and $D_{(1)}f$ are f itself and its global derivative respectively.

Remark 3.1. Using this fact, all the results contained in [6] which involve $D^1 f$ derivatives can be applied also to $D_{(1)}f$ derivatives.

If $D_{(1)}f$ exists, the *Peano derivative of order* 2 in E is

$$D_{(2)}f(x) := (g) \lim_{h \to 0} \frac{2}{h^2} \left\{ f(x+h) - f(x) - h\, f'(x) \right\}, \quad x \in E, \tag{3}$$

when it exists. Moreover, we define also the Peano-Dini derivatives by setting

$$\overline{D}_{(2)}f(x) := (g) \limsup_{h \to 0} \frac{2}{h^2} \left\{ f(x+h) - f(x) - h\, f'(x) \right\}, \quad x \in E; \tag{4}$$

$$\underline{D}_{(2)}f(x) := (g) \liminf_{h \to 0} \frac{2}{h^2} \left\{ f(x+h) - f(x) - h\, f'(x) \right\}, \quad x \in E. \tag{5}$$

The Riemann symmetric and the Peano derivatives are in general different. It is well known in the real case. See, for example, the Heaviside function

$$H(x) := \begin{cases} 1 & \text{if} \quad 0 < x \le 1, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad -1 \le x < 0. \end{cases}$$

For this function $D^2 H(x) = 0$ in $[-1, 1]$. However, H is not continuous at 0, and, a fortiori, H is not differentiable at 0. According to our terminology it has not global second Peano derivative in $[-1, 1]$ (see also [4]).

In the following proposition we identify $\phi(x, h)$ with the corresponding element of $\mathcal{C}_\infty(\Omega)$, where Ω is a compact extremally disconnected topological space as in the Maeda-Ogasawara-Vulikh representation theorem (see [2, Theorem 3]), and $C_\infty(\Omega)$ is the class of all continuous extended real-valued functions, assuming the values $\pm\infty$ at most on a nowhere dense subset of Ω.

In this terms we have the following result:

Proposition 3.2. *If* $(g) \liminf_{h \to 0} \phi(x, h) \ge \underline{\phi}(x)$ $\quad ((g) \limsup_{h \to 0} \phi(x, h) \le \overline{\phi}(x))$ *in* E, *then*

$$\liminf_{h \to 0} [\phi(x, h)(\omega)] \ge \underline{\phi}(x)(\omega) \quad (\limsup_{h \to 0} [\phi(x, h)(\omega)] \le \overline{\phi}(x)(\omega)) \ \text{ in } E$$

for ω *in the complement of a meager subset of* Ω.

Proof. We prove the assertion only in the case of the liminf, since the case of the limsup is analogous. By [6, Proposition 3.4], there is an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ with

$$\phi(x,h) \geq \underline{\phi}(x) - p_\gamma \tag{6}$$

for each $\gamma \in \Gamma$, $x \in E$ and whenever $0 < |h| \leq \gamma(x)$, and so

$$\phi(x,h)(\omega) \geq \underline{\phi}(x)(\omega) - p_\gamma(\omega)$$

for any $\omega \in \Omega$, $\gamma \in \Gamma$, $x \in E$ and $0 < |h| \leq \gamma(x)$. The assertion follows again by [6, Proposition 3.4], since, by the Maeda-Ogasawara-Vulikh representation theorem (see [2]), the net $(p_\gamma(\omega))_{\gamma\in\Gamma}$ is an (o)-net in the complement of a meager subset of Ω. $\square$

We now prove the following version of the Taylor formula.

Theorem 3.3. *Let $f : [a,b] \to R$ have (g)-derivative of order 2 in $[a,b]$. Then*

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2}\,f''(x) + o(h^2) \ \textit{in}\ [a,b]. \tag{7}$$

Proof. By hypothesis there is an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that

$$|f'(x+h) - f'(x) - h\,f''(x)| \leq |h|\,p_\gamma$$

whenever $\gamma \in \Gamma$, $x \in [a,b]$, $0 < |h| \leq \gamma(x)$. The (g)-derivative f' is supposed to be (g)-differentiable and so it is continuous and hence Riemann integrable. By the Fundamental Theorem of Calculus applied to the Riemann integral of f' ([3, Theorem 4.28]) we get, for such γ, x and $h > 0$,

$$\left| \int_0^h f'(x+\tau)\,d\tau - hf'(x) - \left(\int_0^h \tau d\tau\right) f''(x) \right| \leq \left(\int_0^h \tau d\tau\right) p_\gamma,$$

that is

$$\left| f(x+h) - f(x) - hf'(x) - \frac{h^2}{2} f''(x) \right| \leq \frac{h^2}{2}\,p_\gamma.$$

The case $h < 0$ is analogous. This completes the proof. $\square$

The definition of major functions is analogous to the corresponding one given in [6, Definition 6.2], with the difference that we deal with the Peano-Dini derivatives rather than with the Riemann-Dini symmetric derivatives and that here we do not require boundedness of the (g)-derivatives of order 1.

Definition 3.4. Given $f : [a,b] \to R$, we say that $\Psi : [a,b] \to R$ is a *major function of order* 1 (2) for f if it is (g)-continuous ((g)-differentiable) in $[a,b]$ and

3.4.1) $\Psi(a) = 0$ ($\Psi(a) = \Psi'(a) = 0$);

3.4.2) $(g) \liminf\limits_{h\to 0} \dfrac{\Psi(x+h) - \Psi(x)}{h} \geq f(x)$ ($\underline{D}_{(2)}\Psi(x) \geq f(x)$) nearly everywhere in $]a,b[$.

Let $k = 1, 2$. A function Φ is a *minor function of order k for f* if $-\Phi$ is a major function of order k for $-f$.

The following proposition is useful in the sequel.

Proposition 3.5. *Let $T : [a,b] \to R$, $T = \Psi - \Phi$, where Ψ and Φ are a major and minor function of f of order k, respectively. Then T is l-convex, $0 \le l \le k$.*

Proof. The proof proceeds as the one of [6, Proposition 6.3], suitably modified. □

We now introduce the Perron integral of order 1 and 2 based on the *Peano-Dini* derivatives.

Definition 3.6. Let $k = 1, 2$. A function $f : [a,b] \to R$ is said to be *Perron integrable of order k* (shortly *P^k-integrable*) in $[a,b]$, if f has both major and minor functions of order k and

$$\inf_{\Psi \in \mathcal{G}_k} \Psi(b) = \sup_{\Phi \in \mathcal{K}_k} \Phi(b) \in R,$$

where $\mathcal{G}_k$ and $\mathcal{K}_k$ denote the class of all major and minor functions of order k for f, respectively. The above common value is called *P^k-integral* of f on $[a,b]$ and is denoted by

$$(P^k)\int_a^b f(x)\,dx.$$

Remark 3.7. The correctness of the above definition follows from Proposition 3.5 which implies that for any function f the inequality $\inf_{\Psi \in \mathcal{G}_k} \Psi(b) \ge \sup_{\Phi \in \mathcal{K}_k} \Phi(b)$ holds. If $\inf_{\Psi \in \mathcal{G}_k} \Psi(b) = \sup_{\Phi \in \mathcal{K}_k} \Phi(b)$, then for all $x \in [a,b]$ we have: $\inf_{\Psi \in \mathcal{G}_k} \Psi(x) = \sup_{\Phi \in \mathcal{K}_k} \Phi(x)$, since $\Psi - \Phi$ is increasing in $[a,b]$. Thus the function $I_k : [a,b] \to R$ is defined by

$$I_k(x) := \inf_{\Psi \in \mathcal{G}_k} \Psi(x) = \sup_{\Phi \in \mathcal{K}_k} \Phi(x) = (P^k)\int_a^x f(t)\,dt$$

and is called the *P^k-integral function* associated to f. The value $I_k(b)$ of course coincides with the value of the P^k-integral of f on $[a,b]$.

Proposition 3.8. *If $\Phi \in \mathcal{K}_2, \Psi \in \mathcal{G}_2$, then $I_2 - \Phi$ and $\Psi - I_2$ are R_0^+-valued j-convex functions, $j = 0, 1, 2$.*

Proof. The proof of the 2-convexity is the same as the one given in [7], since it does not depend on the type of the derivative (Riemann symmetric or Peano) we are using. □

4 Comparison with other integrals

First of all we note that every Riemann integrable (Ri-integrable) (see [5]) function $f : [a,b] \to R$ is P^1-integrable, and its Riemann integral coincides with the Perron integral, i.e,

$$(Ri)\int_a^x f(t)\,dt = I_1(x) \quad \text{for all} \quad x \in [a,b].$$

It is an immediate consequence of ([6, Proposition 7.7]), since the Peano and the usual global derivative of order 1 coincide.

Now we compare P^1-integral and P^2-integral (for the real case, see [1, Corollary 2.5]).

Theorem 4.1. *If $f : [a,b] \to R$ is P^1-integrable, then f is P^2-integrable too, and*

$$I_2(b) = (P^2)\int_a^b f = (P^1)\int_a^b \left[(P^1)\int_a^x f(t)\,dt\right]\,dx.$$

Proof. Let ψ be any major function of order 1 for f in $[a,b]$. We claim that its *Ri*-integral function Ψ is a major function of order 2 for f.

Without loss of generality, put $h > 0$. Since ψ is a major function of order 1 for f, by (6) of Proposition 3.2 there exist a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable and an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that

$$\psi(x+t) - \psi(x) \geq t\,f(x) - t\,p_\gamma \tag{8}$$

for all $\gamma \in \Gamma$, $x \in E$ and whenever $0 < t \leq \gamma(x)$. Choose now h with $0 < h \leq \gamma(x)$ and take in both sides of (8) the Riemann integral as t varies between 0 and h. We obtain:

$$\Psi(x+h) - \Psi(x) - h\,\Psi'(x) \geq \frac{h^2}{2} f(x) - \frac{h^2}{2} p_\gamma, \quad x \in E, \tag{9}$$

where Ψ denotes the Riemann integral function associated to ψ. The claim follows from (9). Analogously it is possible to prove that, if ϕ is a minor function of order 1 of f, then its (Riemann) integral function Φ is a minor function of order 2 of f.

We write $\inf_1$ when we deal with the class of major functions of order 2 for f which are the integral function of some major function ψ of order 1 for f. Since $\psi - I_1$ is positive and increasing ([6, Proposition 7.5] and Remark 3.1), we get

$$\begin{aligned} 0 \;&\leq\; \inf_{\Psi\in\mathcal{G}_2} \Psi(x) - \int_a^x I_1(t)\,dt \leq \inf_1 \Psi(x) - \int_a^x I_1(t)\,dt \leq \inf_{\psi\in\mathcal{G}_1}\left[\int_a^x \{\psi(t) - I_1(t)\}\,dt\right] \leq \\ &\leq\; (b-a)\inf_{\psi\in\mathcal{G}_1}\left[\sup_{t\in[a,b]}\{\psi(t) - I_1(t)\}\right] = (b-a)\inf_{\psi\in\mathcal{G}_1}\{\psi(b) - I_1(b)\} = 0. \end{aligned}$$

This shows that

$$\inf_{\Psi\in\mathcal{G}_2} \Psi(x) = \int_a^x I_1(t)\,dt.$$

Similarly we can prove that

$$\sup_{\Phi\in\mathcal{K}_2} \Phi(x) = \int_a^x I_1(t)\,dt$$

From this the assertion follows. $\square$

It is easy to understand that the inclusion of P^1-integral into P^2-integral is a strict one. An example can be given in the real-valued case using the fact that P^1-integral functions are continuous while the derivatives of P^2-integral functions can fail to be continuous. Nevertheless these derivatives are always P^1-integrable as it is seen in the following theorem, whose proof is as in [7, Lemma 1] (indeed it is enough to observe once again that the Peano (g)-derivative of order 1 coincides with the usual (g)-derivative).

Theorem 4.2. *If $f : [a,b] \to R$ is P^2-integrable, then I_2' exists and is P^1-integrable. Moreover, if Ψ (resp. Φ) is a major (minor) function of order 2 for f, then*

$$\Psi'(x) - I_2'(x) \geq 0 \quad (I_2'(x) - \Phi'(x) \geq 0) \qquad \textit{for all } x \in [a,b].$$

Now we discuss the relation between our P^2-integral and the second order Perron integral defined in [6] using the Riemann symmetric derivative. First we note that, as it was already mentioned in Introduction, we have assumed in [6], due to some technical reasons, the boundedness of the derivatives of the major and minor functions of order 2. So, under this assumption, the integral defined in [6] does not cover the P^2-integral of the present paper.

But in general, if we equalize the assumptions on the derivatives of the major and minor functions, by assuming only their existence, as it is done here in Definition 3.6, then the second order Perron integral of [6] defined by the Riemann-Dini symmetric derivatives (let us agree to call this integral, under the above assumption, *P_s^2-integral*) is strictly more general than the one based on the Peano-Dini derivatives. We shall see this below in the real-valued case using some example given in [15].

Example 4.3. We note first that, if a function f is (g)-differentiable in $[a,b]$ (pointwise differentiable everywhere on $[a,b]$, in the real-valued case), then its derivative f' is P^1-integrable on any subinterval of $[a,b]$ and f is the P^1-integral function for f'. Then the expression (3) defining the second order Peano derivative of f can be rewritten in the form

$$\lim_{h\to 0} \frac{2}{h^2} \left\{ \int_x^{x+h} f'(t)\, dt - h\, f'(x) \right\}.$$

This limit in fact defines *Cesàro derivative* (*C-derivative*) of f' (see [1] and [10]), i.e., $D_{(2)}f(x) = CDf'(x)$. We get similar equalities for the upper and the lower derivatives defined by (4) and (5):

$$\overline{D}_{(2)}f(x) = \overline{CD}f'(x), \quad \underline{D}_{(2)}f(x) = \underline{CD}f'(x). \tag{10}$$

Moreover differentiability of f in fact coincides with Cesàro continuity (C-continuity) of f'. This allows us to state that if M is a major function for a function f in $[a,b]$, in the sense of the Burkill Cesàro-Perron integral (CP-integral) of order 1 of [10], then $(P^1)\int_a^x M(t)\,dt$ is a major function of order 2, according to Definition 3.4, for f. In the same way, if m is a CP-minor function of order 1 for f in $[a,b]$, then $(P^1)\int_a^x m(t)\,dt$ is a minor function of order 2 for f. It is clear that if M and m are close to each other, then so are their integrals. This implies that, if f is CP-integrable of order 1 and I_1 is its CP-integral function, then f is also P^2-integrable and $(P^1)\int_a^x I_1(t)\,dt$ is its P^2-integral function.

In the opposite direction, if Ψ and Φ are a major and a minor function of order 2, according to Definition 3.4, for a function f, then by (10) Ψ' and Φ' are a CP-major and a CP-minor function of order 1 of f, respectively. Now we use a fact, proved in [16], that if a function has at least one CP-major function and one CP-minor function, then it is CP-integrable. So P^2-integrability implies CP-integrability of order 1 and if I_2 is the P^2-integral function for f, then I_2' (see Theorem 4.2) is the CP-integral function for f. Hence we can state that our P^2-integral is equivalent, at least in the real-valued case, to the Burkill CP-integral of order 1 of [10], meaning by this that the respective classes of integrable functions coincide and the relation between the integral functions is as it is described above.

We are ready now to use the results of [15] where an example of a *continuous* function F on $[0,1]$ was constructed with the following properties:

(i) $F(0)=0$;

(ii) $f(x) := D_s^2\left(\int_0^x F(t)\,dt\right)$ exists everywhere on $]0,1[$;

(iii) f is integrable in the sense of the general Denjoy integral of order 1 (D-integral, see [14]) with the D-integral function $Q(x) := (D)\int_0^x f(t)\,dt$;

(iv) the difference $F-Q$ is a singular Cantor-type function.

We show now that the function f gives the desired example. First of all, the properties (i) and (ii) show that $\int_0^x F(t)\,dt$ plays the role of both a major and a minor function in the sense of the P_s^2-integral. So f is P_s^2-integrable and $\int_0^x F(t)\,dt$ is its P_s^2-integral function. At the same time this function f cannot be P^2-integrable. Indeed if it were, it would be also CP-integrable of order 1 and F would be its CP-integral function. But it is known (see [16]) that the CP-integral is *compatible* with the D-integral, i.e., if a function is both D- and CP-integrable, then the values of the two integrals coincide. So in our case the D-integral function of f would be F. This contradicts to (iii) and (iv). Therefore the function f is P_s^2-integrable but it is not P^2-integrable.

5 Integration by parts formula for P^2-integral

Our proof of the integration by parts formula is based on Theorems 4.1 and 4.2 and on two technical lemmas.

Lemma 5.1. *Let $f:[a,b]\to R$ be P^2-integrable with integral function I_2 and let Ψ, Φ be a major and a minor function of order 2 for f, respectively. Assume that $g:[a,b]\to R$ is a (g)-differentiable function with (g)-continuous derivative such that $g\geq 0$, $g'\geq 0$ in $[a,b]$. Then functions*

$$\begin{aligned} S(t) &= \Psi(t)g(t) - (P^1)\int_a^t \Psi g', \\ Z(t) &= \Phi(t)g(t) - (P^1)\int_a^t \Phi g', \end{aligned} \qquad t\in[a,b] \tag{11}$$

are a major and a minor function of order 2 for $fg + I_2' g'$, respectively.

Proof. We prove the assertion only for S, since the other case is analogous. Observe that, since $\Psi\cdot g'$ is (g)-continuous, from the Torricelli-Barrow theorem ([3, Proposition 4.37]) and 3.4.1) applied to Ψ' we have:

$$\begin{aligned} &S'(t) = \Psi'(t)\,g(t) + \Psi(t)\,g'(t) - \Psi(t)\,g'(t) = \Psi'(t)\,g(t), \\ &S(a) = S'(a) = 0. \end{aligned} \qquad \text{for all } t\in[a,b]. \tag{12}$$

It remains to prove 3.4.2), i.e., $\underline{D}_{(2)}S(x) \geq f(x)g(x) + I_2'(x)g'(x)$. Analogously as in [6, Theorem 8.5], we fix now $x \in [a,b]$, and define for every $t \in [a,b]$:

$$\begin{aligned}
P(t) &= \Psi(x) + (t-x)\Psi'(x); && (13)\\
\overline{\Psi}(t) &= \Psi(t) - P(t); \quad \overline{\Psi}(x) = \Psi(x) - P(x) = 0; && (14)\\
\overline{S}(t) &= \overline{\Psi}(t)g(t) - (P^1)\!\int_x^t \overline{\Psi}\, g'; && (15)\\
V(t) &= S(t) - \overline{S}(t) = [\Psi(t) - \overline{\Psi}(t)]g(t) - (P^1)\!\int_a^t \Psi g' + (P^1)\!\int_x^t \overline{\Psi}\, g' && (16)\\
&\overset{(14)}{=} P(t)g(t) - (P^1)\!\int_a^t Pg' - (P^1)\!\int_a^x \overline{\Psi} g'.
\end{aligned}$$

Then, by (13) we have $P'(t) = \Psi'(x)$. So for every $t \in [a,b]$ we get:

$$\begin{aligned}
V'(t) &= P'(t)\, g(t) + P(t)g'(t) - P(t)g'(t) = \Psi'(x)\, g(t);\\
V''(t) &= \Psi'(x)\, g'(t); && (17)\\
\overline{S}(x) &= \overline{\Psi}(x)\, g(x) = 0;\\
V(x) &= S(x) - \overline{S}(x) = S(x).
\end{aligned}$$

We now give a "Taylor-type" formula up to the order 2 for the functions $\overline{S}$ and V. To this end, we estimate the last summand on the right hand side of (16).

By (g)-differentiability of Ψ, there exists an (o)-net $(\rho_\gamma)_{\gamma\in\Gamma}$ with the property: for every $\gamma \in \Gamma$ and whenever $x, t \in [a,b]$ with $|t-x| \leq \gamma(x)$ we have

$$|\overline{\Psi}(t)| = |\Psi(t) - \Psi(x) - (t-x)\Psi'(x)| \leq |t-x|\, \rho_\gamma. \tag{18}$$

Moreover, there is $K_0 \in R^+$ such that, for every $\tau \in [a,b]$, $|g'(\tau)| \leq K_0$. Using this, we obtain:

$$\begin{aligned}
\left|(P^1)\!\int_x^{x+h} \overline{\Psi}(\tau)\, g'(\tau)\, d\tau\right| &= \left|(P^1)\!\int_x^{x+h} [\Psi(\tau) - \Psi(x) - (\tau - x)\Psi'(x)]g'(\tau)\, d\tau\right| \leq\\
&\leq \left[(P^1)\!\int_x^{x+h} (\tau - x) d\tau\right] K_0\, \rho_\gamma = \frac{h^2}{2}\, K_0\, \rho_\gamma
\end{aligned}$$

whenever $x \in]a,b[$ and $0 < h \leq \gamma(x)$. Analogously we prove that for such x's and h's we have:

$$\left|(P^1)\!\int_{x-h}^{x} \overline{\Psi}(\tau)\, g'(\tau)\, d\tau\right| \leq \frac{h^2}{2}\, K_0\, \rho_\gamma.$$

Hence,

$$(\overline{S} - \overline{\Psi}\, g)(t) = o((t-x)^2) \text{ in } [a,b]. \tag{19}$$

Furthermore, note that g is Lipschitz, and so it follows that the net $(\rho_\gamma)_{\gamma\in\Gamma}$ in (18) satisfies

$$|\overline{\Psi}(t)|\, |g(t) - g(x)| \leq \widetilde{L}\, |t-x|^2\, \rho_\gamma \tag{20}$$

for any $\gamma \in \Gamma$ and $t \in [a,b]$ with $0 < |t-x| \leq \gamma(x)$, where $\widetilde{L}$ is a suitable positive element of R. Hence, from (19), for all $t \in [a,b]$ we get:

$$\overline{S}(t) = g(x)\overline{\Psi}(t) + o((t-x)^2) = g(x)\,[\Psi(t) - \Psi(x) - (t-x)\Psi'(x)] + o((t-x)^2). \tag{21}$$

Now, using (17), we can apply the Taylor formula to V and obtain, for all $t \in [a,b]$:

$$\begin{aligned} V(t) &= V(x) + (t-x)V'(x) + \frac{(t-x)^2}{2}V''(x) + o((t-x)^2) = \\ &= S(x) + (t-x)\Psi'(x)\,g(x) + \frac{(t-x)^2}{2}\Psi'(x)\,g'(x) + o((t-x)^2). \end{aligned} \tag{22}$$

From (21) and (22) we have

$$\begin{aligned} S(t) &= V(t) + \overline{S}(t) = S(x) + (t-x)\Psi'(x)\,g(x) + \\ &+ \frac{(t-x)^2}{2}\Psi'(x)\,g'(x) + g(x)\,[\Psi(t) - \Psi(x) - (t-x)\Psi'(x)] + o((t-x)^2). \end{aligned} \tag{23}$$

From (12) and (23), for each $t \in [a,b]$, $t \neq x$ we get:

$$\begin{aligned} \frac{2}{(t-x)^2}\,[S(t) - S(x) - (t-x)S'(x)] &= \Psi'(x)\,g'(x) + g(x)\cdot \\ &\cdot \frac{2\,[\Psi(t) - \Psi(x) - (t-x)\Psi'(x)]}{(t-x)^2} + o((t-x)^0). \end{aligned}$$

Taking the (g)-liminf, for any $x \in [a,b]$ we have:

$$\underline{D}_{(2)}S(x) = g(x)\,\underline{D}_{(2)}\Psi(x) + \Psi'(x)g'(x).$$

Since Ψ is a major function of order 2 for f and $g(x) \geq 0$ for every $x \in [a,b]$, there exists a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable and for every $x \in E$ we get, thanks to Theorem 4.2,

$$\underline{D}_{(2)}S(x) \geq f(x)\,g(x) + \Psi'(x)g'(x) \geq f(x)\,g(x) + I_2'(x)\,g'(x),$$

since $g'(x) \geq 0$ in $[a,b]$. From this, since Ψ is a major function of order 2 for f, it follows that S is a major function of order 2 for $fg + I_2'g'$. $\square$

Lemma 5.2. *Let $f : [a,b] \to R$ be P^2-integrable with integral function I_2 and let Ψ, Φ be a major and a minor function of order 2 for f, respectively. Assume that $g : [a,b] \to R$ is a (g)-differentiable function with (g)-continuous derivative such that $g \geq 0$, $g' \geq 0$ in $[a,b]$. Let S and Z be defined by (11). Then*

$$S(b) - I_2(b)g(b) + (P^1)\!\int_a^b I_2 g' \geq 0;$$

$$Z(b) - I_2(b)g(b) + (P^1)\!\int_a^b I_2 g' \leq 0.$$

Proof. Thanks to Lemma 5.1 the functions S and Z defined by (11) are a major and a minor function of order 2 for $fg + I_2' g'$, respectively. By properties of major and minor functions of order 2, we get $S(b) - Z(b) \geq 0$, that is

$$\Psi(b)g(b) - (P^1)\!\!\int_a^b \Psi g' \geq \Phi(b)g(b) - (P^1)\!\!\int_a^b \Phi g',$$

and so

$$[\Psi(b) - \Phi(b)]g(b) - (P^1)\!\!\int_a^b (\Psi - \Phi)g' \geq 0.$$

Therefore,

$$[\Psi(b) - I_2(b)]g(b) + [I_2(b) - \Phi(b)]g(b) - (P^1)\!\!\int_a^b (\Psi - I_2)g' - (P^1)\!\!\int_a^b (I_2 - \Phi)g' \geq 0.$$

Since $I_2 - \Phi$ is positive and increasing (by Proposition 3.8) and $g, g' \geq 0$ in $[a, b]$, we get:

$$[\Psi(b) - I_2(b)]g(b) - (P^1)\!\!\int_a^b (\Psi - I_2)g' \geq -[I_2(b) - \Phi(b)]g(b) + (P^1)\!\!\int_a^b (I_2 - \Phi)g' \geq$$
$$\geq -[I_2(b) - \Phi(b)]g(b).$$

If we take the supremum over $\Phi \in \mathcal{K}_2$, we obtain:

$$[\Psi(b) - I_2(b)]g(b) - (P^1)\!\!\int_a^b (\Psi - I_2)g' \geq \left(\sup_{\Phi \in \mathcal{K}_2} [\Phi(b) - I_2(b)]\right) g(b) = 0.$$

Hence,

$$S(b) - I_2(b)g(b) + (P^1)\!\!\int_a^b I_2 g' = [\Psi(b) - I_2(b)]g(b) - (P^1)\!\!\int_a^b (\Psi - I_2)g' \geq 0,$$

obtaining the first inequality of the assertion. The proof of the other inequality is analogous. □

Finally we are ready to prove the integration by parts formula.

Theorem 5.3. *Let $f : [a, b] \to R$ be P^2-integrable, I_2 be the integral function associated to f, I_2' be its derivative, and let $g : [a, b] \to R$ be (g)-differentiable in $[a, b]$, with Lipschitz derivative g' on $[a, b]$. Then, fg is P^2-integrable, and*

$$(P^2)\!\!\int_a^b fg + (P^1)\!\!\int_a^b \left[(P^1)\!\!\int_a^x I_2'(t)g'(t)dt\right] dx = I_2(b)g(b) - (P^1)\!\!\int_a^b I_2 g'.$$

Proof. First of all, we prove the assertion when $g(x) \geq 0$ and $g'(x) \geq 0$ for each $x \in [a, b]$. In order to prove P^2-integrability of fg, we first show that $fg + I_2' g'$ is P^2-integrable.

Let $\mathcal{F}$ and $\mathcal{S}$ be the classes of all major and minor functions of order 2 for $fg + I_2' g'$ respectively. Let $\mathcal{F}_1$ and $\mathcal{S}_1$ be the subclasses of the following type:

$$S(x) = \Psi(x)g(x) - (P^1)\!\!\int_a^x \Psi g', \quad Z(x) = \Phi(x)g(x) - (P^1)\!\!\int_a^x \Phi g', \quad x \in [a, b], \tag{24}$$

(where Ψ, Φ are major and minor functions of order 2 for f). Note that the inclusions $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{S}_1 \subset \mathcal{S}$ was proved in Lemma 5.1.

Now, fix arbitrarily $\Psi \in \mathcal{G}_2$ and let S be as in (24). For all $x \in [a,b]$ we have

$$S(x) - I_2(x)g(x) + (P^1)\!\int_a^x I_2 g' = \Psi(x)g(x) - I_2(x)g(x) - (P^1)\!\int_a^x [\Psi(t) - I_2(t)]g'(t)dt.$$

From this, by virtue of Lemma 5.2, since $g, g' \geq 0$ in $[a,b]$ and $\Psi - I_2$ is positive and increasing (see also [6, Theorem 7.5]), it follows that

$$\begin{aligned} 0 &\leq \inf_{S\in\mathcal{F}_1}\left(S(b) - I_2(b)g(b) + (P^1)\!\int_a^b I_2 g'\right) \leq \\ &\leq \inf_{\Psi\in\mathcal{G}_2}\left(\Psi(b)g(b) - I_2(b)g(b) - (P^1)\!\int_a^b [\Psi(t) - I_2(t)]g'(t)dt\right) \leq \\ &\leq \inf_{\Psi\in\mathcal{G}_2}(\Psi(b)g(b) - I_2(b)g(b)) = \left(\inf_{\Psi\in\mathcal{G}_2}[\Psi(b) - I_2(b)]\right) g(b) = 0, \end{aligned}$$

thanks to the properties of the integral function I_2 and of g'. So we obtain:

$$\inf_{S\in\mathcal{F}_1}\left(S(b) - I_2(b)g(b) + (P^1)\!\int_a^b I_2 g'\right) = 0,$$

that is

$$\inf_{S\in\mathcal{F}_1} S(b) = I_2(b)g(b) - (P^1)\!\int_a^b I_2 g'.$$

Analogously we can prove that

$$\sup_{Z\in\mathcal{S}_1} Z(b) = I_2(b)g(b) - (P^1)\!\int_a^b I_2 g'.$$

By virtue of the main properties of major and minor functions, we get:

$$0 \leq \inf_{S\in\mathcal{F}} S(b) - \sup_{Z\in\mathcal{S}} Z(b) \leq \inf_{S\in\mathcal{F}_1} S(b) - \sup_{Z\in\mathcal{S}_1} Z(b) = 0,$$

that is

$$\inf_{S\in\mathcal{F}} S(b) = \sup_{Z\in\mathcal{S}} Z(b),$$

getting P^2-integrability of $fg + I_2' g'$ on $[a,b]$, and

$$(P^2)\!\int_a^b (fg + I_2' g') = I_2(b)\, g(b) - (P^1)\!\int_a^b I_2 g'.$$

Now, by Theorem 4.2, I_2' is P^1-integrable and, by the theorem of integration by parts for the P^1-integral ([6, Theorem 8.1]), since g' is Lipschitz, we get that $I_2' g'$ is P^1-integrable too. Observe that, by virtue of Theorem 4.1, $I_2' g'$ is also P^2-integrable and

$$(P^2)\!\int_a^b I_2' g' = (P^1)\!\int_a^b \left[(P^1)\!\int_a^x I_2'(t)\, g'(t)\, dt\right] dx.$$

Thus, by the fact that the class of the P^2-integrable functions is linear, fg is P^2-integrable too, and we get

$$(P^2)\int_a^b fg + (P^1)\int_a^b \left[(P^1)\int_a^x I_2'(t)\, g'(t)\, dt\right] dx = I_2(b)g(b) - (P^1)\int_a^b I_2 g', \tag{25}$$

obtaining the assertion, at least when g and g' are positive on $[a, b]$.

In order to drop the positivity condition, observe that formula (25) holds when g is a constant function (see [6, Proposition 7.2 i)]). From this and the boundedness of g, by linearity, it follows that (25) is true, at least for any function g with Lipschitz positive (g)-derivative in $[a, b]$: indeed, it will be enough to take a constant c^* such that $g(x) \geq c^*$ for all $x \in [a, b]$ and to consider the function g as the sum of $g - c^*$ and c^*.

Now we turn to the general case, giving up the positivity assumption on g'. Since g' is Lipschitz, then, by [13, Theorem 12.4 (iii)], $(g')^+$ and $(g')^-$ are two Lipschitz positive functions. Let G^+, G^- be the integral functions of $(g')^+$, $(g')^-$, respectively. Then, by the Torricelli-Barrow Theorem ([3, Proposition 4.37]), $(G^{\pm})' = (g')^{\pm}$. By the Fundamental Theorem of Calculus ([3, Theorem 4.28]), we get that $g - (G^+ - G^-)$ is a constant. From this, and since (25) is true when the role of g is played by $G^{\pm}$ and whenever g is a constant function, we obtain that (25) holds for g and g' in the general case. This completes the proof of the theorem. □

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