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The Monotone Integral - Part II (**).

Sunto. – *Utilizzando la rappresentazione di uno spazio vettoriale localmente convesso E come limite proiettivo di spazi di Banach si estendono risultati ottenuti in* [7].

Summary. – *Using the representation of a lctvs E as the projective limit of Banach spaces we extend the results given in* [7].

1. – Introduction.

The aim of this paper is to construct a theory of integration for scalar functions with respect to finitely additive strongly bounded measures with values in a complete locally convex topological vector space. In [6] the Bochner and the monotone integral are compared in nuclear spaces and some relations between them are obtained under suitable conditions concerning the boundedness of the Radon-Nikodym derivative of the measure with respect to any Rybakov control.

Since the definition of the monotone integral is stronger than the definition of Bochner integral, in [7] the authoresses gave another definition of monotone integrability which involves the Mc-Shane integral, in a Banach space X. More precisely, if $m : \Sigma \to X$ is a finitely additive bounded measure and λ is the Lebesgue measure, a measurable function

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$f : \Omega \to \mathbb{R}_0^+$ is integrable in the monotone sense (($\star$)-integrable) if for every $E \in \Sigma$, there exists an element $w^E \in X$ such that for every $\varepsilon > 0$ there exists a gauge $\Delta(\varepsilon)$ (which must be the same for every $E \in \Sigma$) such that

$$\limsup_{n \to \infty} \left| w^E - \sum_{i=1}^{n} \lambda(T_i)\, \phi^E(t_i) \right| \leqslant \varepsilon$$

for every generalized McShane partition $(T_i, t_i)_i$ subordinate to $\Delta(\varepsilon)$ as defined in [7].

This new definition allows us to obtain the equivalence between Bochner and ($\star$) integrals when one integrates with respect to measures taking values in Banach spaces. This paper is the natural continuation of [7]; here in fact the measures with respect to which we integrate take their values in complete locally convex topological vector spaces and the equivalence obtained in [7] follows from the representation of a lctvs E as the projective limit of Banach spaces. Moreover we introduce the weakly ($\frown$)-integral and we compare it with the previous integrals. We observe that in this case to obtain the equivalence between $L^{1,\star}(m)$ and $(w) - \widehat{L}^1(m)$ we need a definition of Groetendieck integrability, while in [5, 6] the Bochner integrability (in which the defining sequence does not depend on $\alpha \in I$) was necessary.

2. – Definitions of the integrals and their properties.

Let E be a complete locally convex space, $m : \Sigma \to E$ a finitely additive strongly bounded measure.

Every complete locally convex space E is isomorphic to the projective limit of a family of Banach spaces; this family can be chosen such that its cardinality equals the cardinality of a given 0-neighbourhood basis in E (see [9]).

Let $\{U_\alpha : \alpha \in I\}$ be a basis of convex and circled neighbourhoods of 0 in E. We say that $\alpha \leqslant \beta$ if $U_\beta \subset U_\alpha$. If p_α is the gauge of U_α, we can form the projective limit $E = pjl(E_\alpha, g_{\alpha,\beta})$, where $E_\alpha = E_{U_\alpha}$ is the complete Banach space defined as $E_\alpha = E/V_\alpha$, where $V_\alpha = p_\alpha^{-1}(0)$, and $g_{\alpha,\beta}$ is a continuous linear map of E_β into E_α defined by $g_{\alpha,\beta}([x]_\beta) = [x]_\alpha$, for every $\alpha \leqslant \beta$, where $[x]_\alpha$ denotes the equivalence class of the element x with respect to ker (p_α). If $\alpha \leqslant \beta$ then $p_\alpha \leqslant p_\beta$.

We denote by $m_\alpha : \Sigma \to E_\alpha$ the bounded finitely additive measure defined by $m_\alpha(B) = [m(B)]_\alpha$ for every $\alpha \in I$ and for every $B \in \Sigma$. For every $\alpha \in I$, $\Sigma_{\star,\alpha}$ is the σ-algebra generated by Σ and all m_α-null sets.

We refer to [5] for the notations and definitions relative to each m_α and to [7] for the notion of Mc Shane integral.

Note that, since m is bounded, for every $\alpha \in I$ the semivariation $\|m_\alpha\|$ is bounded as well, and for every $\alpha < \beta$ one easily obtains that $\|m_\alpha\| \leq \|m_\beta\|$.

We consider the following definition of integrability:

DEFINITION 2.1. Let $f : \Omega \to \mathbb{R}$ be a Σ-measurable function. Then f is *m-integrable* iff for every $\alpha \in I$ there exists a sequence of simple functions $(f_n^\alpha)_n$ such that:

(i) $(f_n^\alpha)_n \|m_\alpha\|$-converges to f, i.e. f is measurable by seminorms;

(ii) for every $F \in \Sigma$ there exists a $y_F \in E$ such that $\lim_{n \to \infty} p_\alpha(\int_F f_n^\alpha dm - y_F) = 0$, for every $\alpha \in I$.

In this case we set

$$y_F = \int_F f dm .$$

We denote by $L^1(m)$ the space of m-integrable functions.

We observe that if E is a Banach space, f is m-integrable iff f is Bochner integrable. So for every $\alpha \in I$ f is Bochner integrable with respect to m_α.

In [7] we have introduced the following definition in the case of a Banach space:

DEFINITION 2.2. Let Y be a Banach space, $m: \Sigma \to Y$ be a finitely additive bounded measure and λ be the Lebesgue measure. A measurable function $f : \Omega \to \mathbb{R}_0^+$ is integrable in the monotone sense ((⋆)-*integrable*) if for every $E \in \Sigma$, there exists an element $w^E \in Y$ such that for every $\varepsilon > 0$ there exists a gauge $\Delta(\varepsilon)$ (which must be the same for every $E \in \Sigma$) such that

$$\limsup_{n \to \infty} \left| w^E - \sum_{i=1}^{n} \lambda(T_i)\, \phi^E(t_i) \right| \leq \varepsilon$$

for every generalized McShane partition $(T_i, t_i)_i$ subordinate to $\Delta(\varepsilon)$.

We denote by $\overset{\star}{\int_E} f dm = w^E$ and by $L^{1,\star}(m)$ the space of all (⋆)-integrable functions.

Since for every $\alpha \in I$, E_α is a Banach space and $m_\alpha : \Sigma \to E_\alpha$, defined by $m_\alpha(B) = [m(B)]_\alpha$, is a bounded finitely additive measure we can consider the space $L^{1,*}(m_\alpha)$ and we prove that:

LEMMA 2.1. *If* $f \in \bigcap_{\alpha \in I} L^{1,*}(m_\alpha)$ *then* $\left(\int_F^* f dm_\alpha\right)_{\alpha \in I} \in E$.

PROOF. If $f \in \bigcap_{\beta \in I} L^{1,*}(m_\beta)$, by definition, and Theorem 4.4 of [7], $f \in L^{1,*}(m_\beta) = L^1(m_\beta)$, for every $\beta \in I$. We prove that $\left(\int_F f dm_\alpha\right)_\alpha$ is in E, namely, for every $\alpha < \beta$,

$$g_{\alpha,\beta}\left(\int_F f dm_\beta\right) = \int_F f dm_\alpha .$$

By hypothesis $\int_F f dm_\beta = \lim_{n \to \infty} \int_F f_n^\beta dm_\beta$. Since f_n^β $\|m_\beta\|$-converges to f, f is $\Sigma_{*,\beta}$-measurable. So we have

$$\{x \in E : |f(x) - f_n^\beta(x)| > t\} \in \Sigma_{*,\beta} \subset \Sigma_{*,\alpha}$$

and

$$\|m_\alpha\|(|f - f_n^\beta| > t) \leq \|m_\beta\|(|f - f_n^\beta| > t).$$

Therefore f_n^β $\|m_\alpha\|$-converges to f. Since $(\int_F f_n^\beta dm_\beta)_n$ converges in E_β, it is Cauchy in E_β; for every $\varepsilon > 0$ there exists $\overline{n} \in \mathbb{N}$ such that for every $r, s > \overline{n}$

$$p_\beta\left(\int_F (f_r^\beta - f_s^\beta)\, dm_\beta\right) < \varepsilon ,$$

and so $(\int_F f_n^\beta dm_\alpha)_n$ is Cauchy in E_α, since

$$\left|\int_F (f_r^\beta - f_s^\beta)\, dm_\alpha\right|_\alpha = p_\alpha\left(\int_F (f_r^\beta - f_s^\beta)\, dm\right) \leq p_\beta\left(\int_F (f_r^\beta - f_s^\beta)\, dm\right) =$$

$$= \left|\int_F (f_r^\beta - f_s^\beta)\, dm_\beta\right|_\beta .$$

Thus f is m_α-integrable since the integral does not depend on the defining sequence $(f_n^\gamma)_n$, and we obtain

$$g_{\alpha,\beta}\left(\int_F f_n^\beta dm_\beta\right) = \int_F f_n^\beta dm_\alpha$$

and the sequence on the left hand side converges to $\int_F fdm_\alpha$. Therefore we have

$$g_{\alpha,\beta}\left(\int_F fdm_\beta\right)=g_{\alpha,\beta}\left(\lim_{n\to\infty}\int_F f_n^\beta\,dm_\beta\right)=\lim_{n\to\infty}g_{\alpha,\beta}\left(\int_F f_n^\beta\,dm_\beta\right)=$$

$$=\lim_{n\to\infty}\int_F f_n^\beta\,dm_\alpha=\int_F fdm_\alpha.$$

Thus $(\int_F fdm_\alpha)_\alpha\in E$, for every $F\in\Sigma$ and

$$g_{\alpha,\beta}\left(\overset{\star}{\int_F} fdm_\beta\right)=g_{\alpha,\beta}\left(\int_F fdm_\beta\right)=\int_F fdm_\alpha=\overset{\star}{\int_F} fdm_\alpha.$$

Now, using the projective structure of E we define the ($\star$)-integral for a non negative function with respect to a finitely additive strongly bounded measure with values in a complete locally convex space.

DEFINITION 2.3. $f:\Omega\to\mathrm{R}_0^+$ is ($\star$)-*integrable* with respect to m iff f is ($\star$)-integrable with respect to m_α, for every $\alpha\in I$. In this case we set

$$\overset{\star}{\int_F} fdm=\left(\overset{\star}{\int_F} fdm_\alpha\right)_{\alpha\in I}$$

for every $F\in\Sigma$.

We denote by $L^{1,\star}(m)$ the space of all ($\star$)-integrable functions.

In [8] the following definition is given:

DEFINITION 2.4. $f:\Omega\to\mathrm{R}_0^+$ is *weakly* ($\frown$)-*integrable* if $\phi:t\mapsto m(f>t)$ is Pettis-integrable and if $t\mapsto|x^*m|(f>t)\in L^1(\mathrm{R}_0^+)$, for every $x^*\in E^*$. Note that if f is *weakly* ($\frown$)-*integrable*, then for every $F\in\Sigma$, $f\cdot 1_F$ is *weakly* ($\frown$)-*integrable*. In this case we set

$$\overset{\frown}{\int_F} fdm=(P)\int_0^{+\infty} m(f\cdot 1_F>t)\,dt\,.$$

If f is real valued we say that f is *weakly* ($\frown$)-*integrable* iff f^+,f^- are *weakly* ($\frown$)-*integrable*. We denote by $w-\widehat{L}^1(m)$ the space of all weakly ($\frown$)-integrable functions.

3. – Comparison.

PROPOSITION 3.1. *Let $f : \Omega \to \mathbb{R}_0^+$ be a measurable function. Then f is m-integrable iff f is m_α-integrable, for every $\alpha \in I$.*

PROOF. If $f \in L^1(m)$ then there exists a sequence of simple functions $(f_n^\alpha)_n$ such that $f_n^\alpha \| m_\alpha \|$-converges to f and, for evey $F \in \Sigma$, there exists $y_F \in E$ such that for every $\alpha \in I$

$$\lim_{n \to \infty} p_\alpha\left(\int_F f_n^\alpha \, dm - y_F \right) = 0 .$$

Then,

$$p_\alpha\left(\int_F f_n^\alpha \, dm - y_F \right) = p_\alpha\left(\left[\int_F f_n^\alpha \, dm - y_F \right]_\alpha \right) = $$
$$= p_\alpha\left(\left[\int_F f_n^\alpha \, dm \right]_\alpha - [y_F]_\alpha \right) = p_\alpha\left(\int_F f_n^\alpha \, dm_\alpha - [y_F]_\alpha \right).$$

So $f \in \bigcap_{\alpha \in I} L^1(m_\alpha)$.

Viceversa, if $f \in \bigcap_{\alpha \in I} L^1(m_\alpha)$, then, for every $\alpha \in I$, there exists a sequence of simple functions $(f_n^\alpha)_n$ which $\| m_\alpha \|$-converges to f, and such that, for every $F \in \Sigma$, $(\int_\bullet f_n \, dm_\alpha)_n$ converges in E_α. Fix $F \in \Sigma$. By Lemma 2, and by Theorem 4.4 of [7], $(\int_F f dm_\alpha)_\alpha$ is in E. Let $y_F \in E$ be such that for every $\alpha \in I$, $[y_F]_\alpha = \int_F f dm_\alpha$, it only remains to prove that for every $\alpha \in I$, $\lim_{n \to \infty} p_\alpha(y_F - \int_F f_n^\alpha dm) = 0$. But

$$p_\alpha\left(y_F - \int_F f_n^\alpha \, dm \right) = \left\| \left[y_F - \int_F f_n^\alpha \, dm \right]_\alpha \right\|_\alpha =$$
$$= \left\| [y_F]_\alpha - \left[\int_F f_n^\alpha \, dm \right]_\alpha \right\|_\alpha = \left\| \int_F f dm_\alpha - \int_F f_n^\alpha \, dm_\alpha \right\|_\alpha$$

and so $f \in L^1(m)$.

COROLLARY 3.1. *Let $f : \Omega \to \mathbb{R}_0^+$ be a measurable function. Then f is m-integrable iff f is $(\star)$-integrable with respect to m and the two integrals agree.*

PROOF. It follows from Definitions 2.1 and 2.4, from Lemma 2.3 and from the projective structure of E.

Now we want to compare the Bochner integral and the $(\star)$ integral with the weak $(\frown)$-integral. In order to do this we shall need a preliminary Lemma.

LEMMA 3.2. *Let* $\alpha \in I$ *and let* $x_\alpha^* \in E_\alpha^*$. *Then* x^* *defined as* $x^*(x) = x_\alpha^*([x]_\alpha)$, *belongs to* E^*.

PROOF. We have only to prove that $x^* = x_\alpha^* pr_\alpha : E \to E_\alpha \to \mathbb{R}$ is continuous at zero. Since $x_\alpha^* \in E_\alpha^*$ then for every $\varepsilon > 0$ there exists $\delta(\varepsilon, \alpha) > 0$ such that for every $y \in E_\alpha$ with $\|y\|_\alpha \leq \delta$ then $|x_\alpha^*(y)| < \varepsilon$.

Let $V_\varepsilon^\alpha = \{x \in E : p_\alpha(x) \leq \delta(\varepsilon, \alpha)\}$. Let $x \in V_\varepsilon^\alpha$. Then $[x]_\alpha \in E_\alpha$ and $p_\alpha([x]_\alpha) \leq \delta(\varepsilon, \alpha)$. So

$$|x^*(x)| = |x_\alpha^*([x]_\alpha)| < \varepsilon .$$

PROPOSITION 3.2. *Let* f *be weakly* $(\frown)$*-integrable with respect to* m. *Then* f *is weakly* $(\frown)$*-integrable with respect to* m_α *for every* $\alpha \in I$.

PROOF. Let $\alpha \in I$ be fixed. Let $x_\alpha^* \in E_\alpha^*$.

By Lemma 3.3 $x^* = x_\alpha^* pr_\alpha \in E^*$. Thus we have

$$x^*(m) = x_\alpha^*([m]_\alpha) = x_\alpha^*(m_\alpha).$$

Hence

$$|x^*(m)|(f > t) = |x_\alpha^*(m_\alpha)|(f > t).$$

Fix $F \in \Sigma$ and set $y_F = (w) - \widehat{\int_F} f dm \in E$. Then

$$x^*(y_F) = \int_0^\infty x^*(m)(f \cdot 1_F > t)\, dt = \int_0^\infty x_\alpha^*(m_\alpha)(f \cdot 1_F > t)\, dt .$$

Therefore f is weakly $(\frown)$-integrable with respect to m_α, for every $\alpha \in I$ and

$$\widehat{\int_\bullet} f dm_\alpha = \left[\widehat{\int_\bullet} dm \right]_\alpha .$$

Now we want to compare $L^{1,\star}(m)$ and $w - \widehat{L}^1(m)$.

First we compare them in a Banach space. Let Y be a Banach space and let $\mu : \Sigma \to Y$ be a finitely additive strongly bounded measure. We set $\phi^E(t) = \mu(\{x \in E : f(x) > t\})$ and $\phi_n^E(t) = \mu(\{x \in E : f_n(x) > t\})$.

THEOREM 3.1. *If* $f: \Omega \to \mathbb{R}_0^+$ *is a measurable function such that there exists a sequence of simple functions* $(f_n)_n$, $f_n \leq f$ *such that*

i) f_n $\|\mu\|$*-converges to* f,

ii) $(w) - \lim_{n\to\infty} \phi_n^E(t) = \phi^E(t)$,

iii) $\lim_{n\to\infty} \int_B \phi_n^E(t)\,dt = \int_B \phi^E(t)\,dt$ *exists in* Y, *for the weak topology,*

for every $B \in \mathcal{B}$ *then* f *is* $(\star)$*-integrable with respect to* μ.

PROOF. The proof is the same as in Theorem 4.8 of [7], in fact in order to prove that $\int_\bullet \mu(x \in E : f_n > t)\,dt : \mathcal{B} \to Y$ is countably additive, for every $E \in \Sigma$, the condition iii) is enough.

THEOREM 3.2. *Let* $f: \Omega \to \mathbb{R}_0^+$ *be a measurable function such that* $\lim_{t\to\infty} \|\mu\|(f > t) = 0$. *Then* f *is* $(\star)$*-integrable with respect to* μ *iff* f *is weakly* $(\frown)$*-integrable with respect to* μ.

PROOF. Suppose that f is $(\star)$-integrable; then for every $E \in \Sigma$, $\phi^E(t)$ is McShane-integrable and so by Theorem 1.Q of [4] $\phi^E(t)$ is Pettis-integrable and the two integrals agree.

It remains to prove that for every $x^\star \in Y^\star$ $|x^\star\mu|(f > t) \in L^1(\mathbb{R}_0^+)$.

Since $\phi^E(t)$ is Pettis-integrable $\int_0^{+\infty} |x^\star\mu|(\{\omega \in E : f(\omega) > t\})\,dt < +\infty$ and by Theorems 3.5, 3.6 of [2] f is integrable with respect to $|x^\star\mu|$.

Suppose now that f is weakly $(\frown)$-integrable. By hypothesis for every $E \in \Sigma$ ϕ^E is Pettis-integrable, namely for every $E \in \Sigma$ there exists $w_E \in X$ such that $\int_0^{+\infty} x^\star\mu(\{\omega \in E : f(\omega) > t\})\,dt = x^\star w_E$ and $|x^\star\mu|(f > t) \in$ $\in L^1(\mathbb{R}_0^+)$.

Since f is measurable by Proposition 3.2 of [7] ϕ is totally measurable. As in Proposition 3.2 of [7] there exists a sequence of simple functions $(f_n)_n$ with $f_n \leq f$ for every $n \in \mathbb{N}$ and f_n converges to f μ-a.e.

Let $\phi_n(t) = \mu\{\omega \in \Omega : f_n(\omega) > t\}$. Then $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ λ-a.e. and

$$|x^\star\mu|\{\omega \in \Omega : f_n(\omega) > t\} \leq |x^\star\mu|\{\omega \in \Omega : f(\omega) > t\}.$$

Let $E \in \Sigma$ be fixed. We shall prove that the limit $\lim_{n\to\infty} \int_B \phi_n^E(t)\,dt$ exists in Y, for the weak topology, for every $B \in \mathcal{B}$.

By Proposition 4.2 of [7] ϕ_n^E is Bochner-integrable, for every $n \in \mathbb{N}$ and so $x^* \phi^E - x^* \phi_n^E \in L^1(\mathbb{R}_0^+)$. Moreover

$$|x^* \phi^E - x^* \phi_n^E| \leqslant |x^* \phi^E| + |x^* \phi_n^E| \leqslant |x^* \mu| \{\omega \in \Omega : f_n(\omega) > t\} +$$
$$+ |x^* \mu| \{\omega \in \Omega : f(\omega) > t\} \leqslant 2 |x^* \mu| (f > t) \in L^1(\mathbb{R}_0^+)$$

and so we have

$$\limsup_{n \to \infty} \int_B |x^* \phi^E(t) - x^* \phi_n^E(t)| dt \leqslant$$
$$\leqslant \int_B \limsup_{n \to \infty} |x^* \phi^E(t) - x^* \phi_n^E(t)| dt \leqslant \int_0^{+\infty} 2 |x^* \mu| (f > t)\, dt .$$

Since $\lim_{n \to \infty} |x^* \phi^E(t) - x^* \phi_n^E(t)| = 0$ λ-a.e. it follows that

$$\lim_{n \to \infty} \int_B |x^* \phi^E(t) - x^* \phi_n^E(t)| dt = 0$$

and so

$$\lim_{n \to \infty} \left| \int_B x^* \phi_n^E(t)\, dt - x^* w_E \right| = \lim_{n \to \infty} \left| \int_B x^* \phi_n^E(t)\, dt - \int_B x^* \phi^E(t)\, dt \right| \leqslant$$
$$\leqslant \lim_{n \to \infty} \int_B |x^* \phi_n^E(t) - x^* \phi^E(t)| dt = 0 .$$

Thus we have proved the existence of the limit $\lim_{n \to \infty} \int_B \phi_n^E(t)\, dt$ in Y, for the weak topology, for every $B \in \mathcal{B}$.

By Theorem 3.5 f is $(\star)$-integrable and the two integrals agree.

Corollary 3.2. *Let $f : \Omega \to \mathbb{R}_0^+$ be a measurable function such that $\lim_{t \to \infty} \|m_a\|(f > t) = 0$. Then $L^1(m) = L^{1,\star}(m) = w - \widehat{L}^1(m)$.*

Proof. It follows from Corollary 3.2, Theorem 3.6, where $\mu = m_a$, and Proposition 3.4.

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